and/or equations in the system—the more difficult it often is to solve the system. For instance, suppose we needed to find the partial fraction decomposition of

$$\frac{1}{(x^2+1)(x^2+4)},$$

which, as you saw in calculus, is used to integrate this expression. (In our study of the Laplace transform in Chapter 7, we will see another place where finding partial fraction decompositions of expressions such as this arises.) This partial fraction decomposition has the form

$$\frac{1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4},$$

and finding it involves solving a system of four linear equations in the four unknowns A, B, C, and D, which takes more time and effort to solve than the problem with the boat. There is no limit to the size of linear systems that arise in practice. It is not unheard of to encounter systems of linear equations with tens, hundreds, or even thousands of unknowns and equations.

The larger the linear system, the easier it is to get lost in your work if you are not careful. Because of this, we are going to begin this chapter by showing you a systematic way of solving linear systems of equations so that, if you follow this approach, you will always be led to the correct solutions of a given linear system. Our approach will involve representing linear systems of equations by a type of expression called a matrix. After you have seen this particular use of matrices (it will be just one of many more to come) in Section 1.1, we will go on to study matrices in their own right in the rest of this chapter. We begin with a discussion of some of the basics.

1.1 Systems of Linear Equations

A **linear equation** in the variables or unknowns $x_1, x_2, \ldots, x_n$ is an equation that can be written in the form

$$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b}$$

where $a_1, a_2, \ldots, a_n, b$ are constants. For instance,

$$2x - 3y = 1$$

is a linear equation in the variables x and y,

$$3x - y + 2z = 8$$

is a linear equation in the variables x, y, and z, and

$$-x_1 + 5x_2 - \pi x_3 + \sqrt{2}x_4 - 9x_5 = e^2$$

is a linear equation in the variables x_1, x_2, x_3, x_4, and x_5. The graph of a linear equation in two variables such as $2x - 3y = 1$ is a line in the xy-plane, and the graph of a linear equation in three variables such as $3x - y + 2z = 8$ is a plane in 3-space.

When considered together, a collection of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is called a **system of linear equations.** For instance,

$$\begin{aligned} x - y + z &= 0 \\ 2x - 3y + 4z &= -2 \\ -2x - y + z &= 7 \end{aligned}$$

is a system of three linear equations in three variables.

A **solution** to a system of equations with variables $x_1, x_2, \ldots, x_n$ consists of values of $x_1, x_2, \ldots, x_n$ that satisfy each equation in the system. From your first algebra course you should recall that the solutions to a system of two linear equations in x and y,

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2, \end{aligned}$$

are the points at which the graphs of the lines given by these two equations intersect. Consequently, such a system will have exactly one solution if the graphs intersect in a single point, will have infinitely many solutions if the graphs are the same line, and will have no solution if the graphs are parallel. As we shall see, this in fact holds for all systems of linear equations; that is, a linear system either has exactly one solution, infinitely many solutions, or no solutions.

The main purpose of this section is to present the **Gauss-Jordan elimination method,**[1] a systematic way for solving systems of linear equations that will always lead us to solutions of the system. The Gauss-Jordan method involves the repeated use of three basic transformations on a system. We shall call the following transformations **elementary operations.**

1. Interchange two equations in the system.
2. Multiply an equation by a nonzero number.
3. Replace an equation by itself plus a multiple of another equation.

Two systems of equations are said to be **equivalent** if they have the same solutions. It is not difficult to see that applying an elementary operation to a system produces an equivalent system.

[1] Named in honor of Karl Friedrich Gauss (1777–1855), who is one of the greatest mathematicians of all time and is often referred to as the "prince of mathematics," and Wilhelm Jordan (1842–1899), a German engineer.

To illustrate the Gauss-Jordan elimination method, consider the system:

$$\begin{aligned} x - y + z &= 0 \\ 2x - 3y + 4z &= -2 \\ -2x - y + z &= 7. \end{aligned}$$

We are going to use elementary operations to transform this system to one of the form

$$\begin{aligned} x \qquad\qquad &= * \\ y \qquad &= * \\ z &= * \end{aligned}$$

where each $*$ is a constant from which we have the solution. To this end, let us first replace the second equation by itself plus -2 times the first equation (or subtracting 2 times the first equation from the second) and replace the third equation by itself plus 2 times the first equation to eliminate x from the second and third equations. (We are doing two elementary operations simultaneously here.) This gives us the system

$$\begin{aligned} x - y + z &= 0 \\ -y + 2z &= -2 \\ -3y + 3z &= 7. \end{aligned}$$

Next, let us use the second equation to eliminate y in the first and third equations by replacing the first equation by itself minus the second equation and replacing the third equation by itself plus -3 times the second equation, obtaining

$$\begin{aligned} x \qquad -z &= 2 \\ -y + 2z &= -2 \\ -3z &= 13. \end{aligned}$$

Now we are going to use the third equation to eliminate z from the first two equations by multiplying the first equation by 3 and then subtracting the third equation from it (we actually are doing two elementary operations here) and multiplying the second equation by 3 and then adding 2 times the third equation to it (here too we are doing two elementary operations). This gives us the system:

$$\begin{aligned} 3x \qquad\qquad &= -7 \\ -3y \qquad &= 20 \\ 3z &= -13. \end{aligned}$$

Finally, dividing the first equation by 3 (or multiplying it by 1/3), dividing the second equation by -3, and dividing the third equation by 3, we have our system in the promised form as

$$\begin{aligned} x \qquad\qquad &= -\frac{7}{3} \\ y \qquad &= -\frac{20}{3} \\ z &= -\frac{13}{3}, \end{aligned}$$

which tells us the solution.

You might notice that we only really need to keep track of the coefficients as we transform our system. To keep track of them, we will indicate a system such as

$$\begin{aligned} x - y + z &= 0 \\ 2x - 3y + 4z &= -2 \\ -2x - y + z &= 7 \end{aligned}$$

by the following array of numbers:

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right].$$

This array is called the **augmented matrix** for the system. The entries appearing to the left of the dashed vertical line are the coefficients of the variables as they appear in the system. This part of the augmented matrix is called the **coefficient matrix** of the system. The numbers to the right of the dashed vertical line are the constants on the right-hand side of the system as they appear in the system. In general, the augmented matrix for the system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right].$$

The portion of the augmented matrix to the left of the dashed line with entries a_{ij} is the coefficient matrix of the system.

Corresponding to the elementary operations for systems of equations are elementary row operations that we perform on the augmented matrix for a linear system. These are as follows.

1. Interchange two rows.[2]
2. Multiply a row by a nonzero number.
3. Replace a row by itself plus a multiple of another row.

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[2] A line of numbers going across the matrix from left to right is called a **row**; a line of numbers going down the matrix is called a **column**.

As our first formal example of this section, we are going to redo the work we did in solving the system

$$\begin{aligned} x - y + z &= 0 \\ 2x - 3y + 4z &= -2 \\ -2x - y + z &= 7 \end{aligned}$$

with augmented matrices.

EXAMPLE 1 Solve the system:

$$\begin{aligned} x - y + z &= 0 \\ 2x - 3y + 4z &= -2 \\ -2x - y + z &= 7. \end{aligned}$$

Solution Our work will consist of four steps. In the first step, we shall use the first row and row operations to make all other entries in the first column zero. In the second step, we shall use the second row to make all other entries in the second column zero. In the third step, we shall use the third row to make all other entries in the third column zero. In the fourth step, we shall make the nonzero entries in the coefficient matrix 1 at which point we will be able to read off our solution. To aid you in following the steps, an expression such as $R_2 - 2R_1$ next to the second row indicates that we are replacing the second row by itself plus -2 times the first row; an expression such as $R_1/3$ next to the first row indicates we are dividing this row by 3. Arrows are used to indicate the progression of our steps.

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 2R_1 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & -3 & 3 & 7 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 3R_2 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -3 & 13 \end{array}\right] \begin{array}{l} 3R_1 - R_3 \\ 3R_2 + 2R_3 \\ \\ \end{array} \to \left[\begin{array}{ccc|c} 3 & 0 & 0 & -7 \\ 0 & -3 & 0 & 20 \\ 0 & 0 & -3 & 13 \end{array}\right] \begin{array}{l} R_1/3 \\ -R_2/3 \\ -R_3/3 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -7/3 \\ 0 & 1 & 0 & -20/3 \\ 0 & 0 & 1 & -13/3 \end{array}\right]$$

The solution is then $x = -7/3$, $y = -20/3$, $z = -13/3$. ●

In Gauss-Jordan elimination, we use elementary row operations on the augmented matrix of the system to transform it so that the final coefficient matrix has a form called **reduced row-echelon form** with the following properties.

1. Any rows of zeros (called **zero rows**) appear at the bottom.
2. The first nonzero entry of a nonzero row is 1 (called a **leading 1**).
3. The leading 1 of a nonzero row appears to the right of the leading 1 of any preceding row.
4. All the other entries of a column containing a leading 1 are zero.

Looking back at Example 1, you will see that the coefficient matrix in our final augmented matrix is in reduced row-echelon form. Once we have the coefficient matrix in reduced row-echelon form, the solutions to the system are easily determined.

Let us do some more examples.

EXAMPLE 2 Solve the system:

$$\begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 1 \\ x_1 + x_2 + x_4 &= 2 \\ x_1 + 2x_2 - 4x_3 &= 1 \\ 2x_1 + x_2 + 2x_3 + 5x_4 &= 1. \end{aligned}$$

Solution We try to proceed as we did in Example 1. Notice, however, that we will have to modify our approach here. The symbol $R_2 \leftrightarrow R_3$ after the first step is used to indicate that we are interchanging the second and third rows.

$$\left[\begin{array}{rrrr|r} 1 & 1 & -1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & -4 & 0 & 1 \\ 2 & 1 & 2 & 5 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - 2R_1 \end{array}$$

$$\to \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & -1 & 4 & 1 & -1 \end{array}\right] \begin{array}{l} \\ R_2 \leftrightarrow R_3 \\ \\ \\ \end{array}$$

$$\to \left[\begin{array}{rrrr|r} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 4 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ \\ R_4 + R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 1 \\ 0 & 1 & -3 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{array}\right] \begin{array}{l} R_1 - 2R_3 \\ R_2 + 3R_3 \\ \\ R_4 - R_3 \end{array} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & -1 \\ 0 & 1 & 0 & -5 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

We now have the coefficient matrix in reduced row-echelon form. Our final augmented matrix represents the system

$$\begin{aligned} x_1 + 6x_4 &= -1 \\ x_2 - 5x_4 &= 3 \\ x_3 - x_4 &= 1 \\ 0 &= -2, \end{aligned}$$

which is equivalent to our original system. Since this last system contains the false equation $0 = -2$, it has no solutions. Hence our original system has no solutions. ●

EXAMPLE 3 Solve the system:

$$\begin{aligned} 2x + 3y - z &= 3 \\ -x - y + 3z &= 0 \\ x + 2y + 2z &= 3 \\ y + 5z &= 3. \end{aligned}$$

Solution We first reduce the augmented matrix for this system so that its coefficient matrix is in reduced row-echelon form.

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 3 \\ -1 & -1 & 3 & 0 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 5 & 3 \end{array}\right] \begin{array}{l} \\ 2R_2 + R_1 \\ 2R_3 - R_1 \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 3 & -1 & 3 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 5 & 3 \end{array}\right] \begin{array}{l} R_1 - 3R_2 \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 0 & -16 & -6 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_1/2 \\ \\ \\ \\ \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -8 & -3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This final augmented matrix represents the equivalent system:

$$x - 8z = -3$$
$$y + 5z = 3$$
$$0 = 0$$
$$0 = 0.$$

Solving the first two equations for x and y in terms of z, we can say that our solutions have the form

$$x = -3 + 8z, \qquad y = 3 - 5z$$

where z is any real number. In particular, we have infinitely many solutions in this example. (Any choice of z gives us a solution. If $z = 0$, we have $x = -3$, $y = 3$, $z = 0$ as a solution; if $z = 1$, we have $x = 5$, $y = -2$, $z = 1$ as a solution; if $z = \sqrt{17}$, we have $x = -3 + 8\sqrt{17}$, $y = 3 - 5\sqrt{17}$, $z = \sqrt{17}$ as a solution; and so on.) In a case such as this, we refer to z as the **free variable** and x and y as the **dependent** variables in our solutions. When specifying our solutions to systems like this, we will follow the convention of using variables that correspond to leading ones as dependent variables and those that do not as free variables. It is not necessary to specify our solutions this way, however. For instance, in this example we could solve for z in terms of x, obtaining

$$z = \frac{x}{8} + \frac{3}{8}$$

and

$$y = 3 - 5z = 3 - 5\left(\frac{x}{8} + \frac{3}{8}\right) = -\frac{5x}{8} + \frac{9}{8},$$

giving us the solutions with x as the free variable and y and z as the dependent variables. ●

EXAMPLE 4 Solve the system:

$$4x_1 - 8x_2 - x_3 + x_4 + 3x_5 = 0$$
$$5x_1 - 10x_2 - x_3 + 2x_4 + 3x_5 = 0$$
$$3x_1 - 6x_2 - x_3 + x_4 + 2x_5 = 0.$$

Solution We again begin by reducing the augmented matrix to the point where its coefficient matrix is in reduced row-echelon form:

$$\left[\begin{array}{ccccc|c} 4 & -8 & -1 & 1 & 3 & 0 \\ 5 & -10 & -1 & 2 & 3 & 0 \\ 3 & -6 & -1 & 1 & 2 & 0 \end{array}\right] \begin{array}{l} \\ 4R_2 - 5R_1 \\ 4R_3 - 3R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccccc|c} 4 & -8 & -1 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ R_3 + R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccccc|c} 4 & -8 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 0 & 4 & -4 & 0 \end{array}\right] \begin{array}{l} R_1/4 \\ \\ R_3/4 \end{array}$$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - 3R_3 \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccccc|c} 1 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right].$$

We now have arrived at the equivalent system

$$\begin{aligned} x_1 - 2x_2 + x_5 &= 0 \\ x_3 &= 0 \\ x_4 - x_5 &= 0, \end{aligned}$$

which has solutions

$$x_1 = 2x_2 - x_5, \qquad x_3 = 0, \qquad x_4 = x_5$$

with x_2 and x_5 as the free variables and x_1 and x_4 as the dependent variables. ●

Systems of equations that have solutions such as those in Examples 1, 3, and 4 are called **consistent systems**; those that do not have solutions as occurred in Example 2 are called **inconsistent systems.** Notice that an inconsistent system is easily recognized once the coefficient matrix of its augmented matrix is put in reduced row-echelon form: There will be a row with zeros in the coefficient matrix with nonzero entry in the right-hand entry of this row. If we do not have this, the system is consistent. Consistent systems break down into two types. Once the coefficient matrix of the augmented matrix is put in reduced row-echelon form, the number of nonzero rows in the coefficient matrix is always less than or equal to the number of columns of the coefficient matrix. (That is, there will never be more nonzero rows than columns when the coefficient matrix is in reduced row-echelon form. Why is this the case?) If there are fewer nonzero rows than columns, as we had in Examples 3 and 4, the system will have infinitely many solutions. If we have as many nonzero rows as columns, as occurred in Example 1, we have exactly one solution. Recall that it was mentioned at the beginning of this section that every system of linear equations either has exactly one solution, infinitely many solutions, or no solutions. Now we can see why this is true.

A system of linear equations that can be written in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned} \tag{1}$$

is called a **homogeneous system.** The system of equations in Example 4 is homogeneous. Notice that

$$x_1 = 0, \quad x_2 = 0, \quad \ldots, \quad x_n = 0$$

is a solution to the homogeneous system in Equations (1). This is called the **trivial solution** of the homogeneous system. Because homogeneous systems always have a trivial solution, they are never inconsistent systems. Homogeneous systems will occur frequently in our future work and we will often be interested in whether such a system has solutions other than the trivial one, which we naturally call **nontrivial solutions.** The system in Example 4 has nontrivial solutions. For instance, we would obtain one (among the infinitely many such nontrivial solutions) by letting $x_2 = 1$ and $x_5 = 2$, in which case we have the nontrivial solution $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 2$, $x_5 = 2$. Actually, we can tell ahead of time that the system in Example 4 has nontrivial solutions. Because this system has fewer equations than variables, the reduced row-echelon form of the coefficient matrix will have fewer nonzero rows than columns and hence must have infinitely many solutions (only one of which is the trivial solution) and consequently must have infinitely many nontrivial solutions. This reasoning applies to any homogeneous system with fewer equations than variables, and hence we have the following theorem.

THEOREM 1.1 A homogeneous system of m linear equations in n variables with $m < n$ has infinitely many nontrivial solutions.

Of course, if a homogeneous system has at least as many equations as variables such as the systems

$$\begin{aligned} x + y + z &= 0 \\ x - y - z &= 0 \\ 2x + y + z &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x + y + z &= 0 \\ x - 2y - z &= 0 \\ 3x - y &= 0 \\ 4x - 3y - z &= 0 \end{aligned}$$

we would have to do some work toward solving these systems before we would be able to see whether they have nontrivial solutions. We shall do this for the second system a bit later.

Gaussian elimination, which is another systematic approach for solving linear systems, is similar to the approach we have been using but does not require that all the other entries of the column containing a leading 1 be zero. That is, it uses row operations to transform the augmented matrix so that the coefficient matrix has the following form:

1. Any zero rows appear at the bottom.

2. The first nonzero entry of a nonzero row is 1.
3. The leading 1 of a nonzero row appears to the right of the leading 1 of any preceding row.

Such a form is called a **row-echelon form** for the coefficient matrix. In essence, we do not eliminate (make zero) entries above the leading 1s in Gaussian elimination. Here is how this approach can be applied to the system in Example 1.

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + 2R_1 \end{array}$$

$$\to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & -3 & 3 & 7 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 3R_2 \end{array}$$

$$\to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -3 & 13 \end{array}\right] \begin{array}{l} \\ -R_2 \\ -R_3/3 \end{array} \to \left[\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -13/3 \end{array}\right]$$

We now have the coefficient matrix in a row-echelon form and use this result to find the solutions. The third row tells us

$$z = -\frac{13}{3}.$$

The values of the remaining variables are found by a process called **back substitution.** From the second row, we have the equation

$$y - 2z = 2$$

from which we can find y:

$$y + \frac{26}{3} = 2$$

$$y = -\frac{20}{3}.$$

Finally, the first row represents the equation

$$x - y + z = 0$$

from which we can find x:

$$x + \frac{20}{3} - \frac{13}{3} = 0$$

$$x = -\frac{7}{3}.$$

On the plus side, Gaussian elimination requires fewer row operations. But on the minus side, the work is sometimes messy when doing the back substitutions. Often, we find ourselves having to deal with fractions even if our original system involves only integers. The back substitutions are also cumbersome to do when dealing with systems that have infinitely many solutions. Try the Gaussian elimination procedure in Example 3 or 4 if you would like to see how it goes.

As a rule we will tend to use Gauss-Jordan elimination when we have to find the solutions to a linear system in this text. Sometimes, however, we will not have to completely solve a system and will use Gaussian elimination since it will involve less work. The next example illustrates an instance of this. In fact, in this example we will not even have to bother completing Gaussian elimination by making the leading entries one.

EXAMPLE 5 Determine the values of a, b, and c so that the system

$$\begin{aligned} x - y + 2z &= a \\ 2x + y - z &= b \\ x + 2y - 3z &= c \end{aligned}$$

has solutions.

Solution We begin doing row operations as follows.

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 2 & 1 & -1 & b \\ 1 & 2 & -3 & c \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 0 & 3 & -5 & b - 2a \\ 0 & 3 & -5 & c - a \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & -1 & 2 & a \\ 0 & 3 & -5 & b - 2a \\ 0 & 0 & 0 & a - b + c \end{array}\right]$$

Now we can see that this system has solutions if and only if a, b, and c satisfy the equation

$$a - b + c = 0.$$

●

Another place where we will sometimes use an abbreviated version of Gaussian elimination is when we are trying to see if a homogeneous system has nontrivial solutions.

EXAMPLE 6 Determine if the system

$$\begin{aligned} 2x + y + z &= 0 \\ x - 2y - z &= 0 \\ 3x - y &= 0 \\ 4x - 3y - z &= 0 \end{aligned}$$

has nontrivial solutions.

Solution Perform row operations:

$$\left[\begin{array}{rrr|r} 2 & 1 & 1 & 0 \\ 1 & -2 & -1 & 0 \\ 3 & -1 & 0 & 0 \\ 4 & -3 & -1 & 0 \end{array}\right] \begin{array}{c} \\ 2R_2 - R_1 \\ 2R_3 - 3R_1 \\ R_4 - 2R_1 \end{array} \rightarrow \left[\begin{array}{rrr|r} 2 & 1 & 1 & 0 \\ 0 & -5 & -3 & 0 \\ 0 & -5 & -3 & 0 \\ 0 & -5 & -3 & 0 \end{array}\right] \begin{array}{c} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{rrr|r} 2 & 1 & 1 & 0 \\ 0 & -5 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

It is now apparent that this system has nontrivial solutions. In fact, you should be able to see this after the first set of row operations. ●

It is not difficult to write computer programs for solving systems of linear equations using the Gauss-Jordan or Gaussian elimination methods. Thus it is not surprising that there are computer software packages for solving systems of linear systems.[3] Maple is one among several available mathematical software packages that can be used to find the solutions of linear systems of equations.

In the preface we mentioned that we will use Maple as our accompanying software package within this text. The use of Maple is at the discretion of your instructor. Some may use it, others may prefer to use a different software package, and yet others may choose to not use any such package (and give an excellent and complete course). For those intructors who wish to use Maple—or for students who are independently interested in gaining some knowledge of its capabilities—we will include occasional remarks about how to use it when we deem it appropriate. On many other occasions we will not include any remarks and will simply provide some exercises asking you to use indicated Maple commands. In these cases, you are expected to look up the command in the Help menu under Topic Search to see how to use it. This is one place where we will include a few remarks to get you started. For those who wish to use the software packages Mathematica or MATLAB, the accompanying *Technology Resource Manual* contains corresponding commands for these software packages.

Here we explain how to use Maple to find solutions to linear systems. One way to do this is to use the *linsolve* command. To use this command in a Maple worksheet, you will first have to load Maple's linear algebra package by typing

```
with(linalg);
```

at the command prompt > and then hitting the enter key. After doing this, you will get a list of Maple's linear algebra commands. To solve the system in Example 1, first enter the coefficient matrix of the system by typing

```
A:= matrix([[1,-1,1],[2,-3,4],[-2,-1,1]]);
```

[3] Often these packages employ methods that are more efficient than Gauss-Jordan or Gaussian elimination, but we will not concern ourselves with these issues in this text.

at the command prompt and then hitting the enter key. (The symbol := is used in Maple for indicating that we are defining A to be the coefficient matrix we type on the right.) The constants on the right-hand side of the system are typed and entered as

```
b:=vector([0,-2,7]);
```

at the command prompt. Finally, type and enter

```
linsolve(A,b);
```

at the commmand prompt and Maple will give us the solution as

$$\left[\frac{-7}{3}, \frac{-20}{3}, \frac{-13}{3}\right]^4.$$

Doing the same set of steps for the system in Example 2 results in no output, indicating there is no solution. Doing them in Example 3 yields the output

$$[-3 + 8\_t_1, 3 - 5\_t_1, \_t_1],$$

which is Maple's way of indicating our solutions in Example 3 with t_1 in place of z. In Example 4, these steps yield

$$[2\_t_1 - \_t_2, \_t_1, 0, \_t_2, \_t_2].$$

EXERCISES 1.1

Solve the systems of equations in Exercises 1–16.

1. $$\begin{aligned} x + y - z &= 0 \\ 2x + 3y - 2z &= 6 \\ x + 2y + 2z &= 10 \end{aligned}$$

2. $$\begin{aligned} 2x + y - 2z &= 0 \\ 2x - y - 2z &= 0 \\ x + 2y - 4z &= 0 \end{aligned}$$

3. $$\begin{aligned} 2x + 3y - 4z &= 3 \\ 2x + 3y - 2z &= 3 \\ 4x + 6y - 2z &= 7 \end{aligned}$$

4. $$\begin{aligned} 3x + y - 2z &= 3 \\ x - 8y - 14z &= -14 \\ x + 2y + z &= 2 \end{aligned}$$

5. $$\begin{aligned} x + 3z &= 0 \\ 2x + y - z &= 0 \\ 4x + y + 5z &= 0 \end{aligned}$$

6. $$\begin{aligned} 2x + 3y + z &= 4 \\ x + 9y - 4z &= 2 \\ x - y + 2z &= 3 \end{aligned}$$

7. $$\begin{aligned} 3x_1 + x_2 - 3x_3 - x_4 &= 6 \\ x_1 + x_2 - 2x_3 + x_4 &= 0 \\ 3x_1 + 2x_2 - 4x_3 + x_4 &= 5 \\ x_1 + 2x_2 - 3x_3 + 3x_4 &= 4 \end{aligned}$$

8. $$\begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 1 \\ x_1 + x_2 - x_3 - x_4 &= -1 \\ x_1 + 2x_2 + x_3 + 2x_4 &= -1 \\ 2x_1 + 2x_2 + x_3 + x_4 &= 2 \end{aligned}$$

9. $$\begin{aligned} x_1 + 2x_2 - 3x_3 + 4x_4 &= 2 \\ 2x_1 - 4x_2 + 6x_3 - 5x_4 &= 10 \\ x_1 - 6x_2 + 9x_3 - 9x_4 &= 8 \\ 3x_1 - 2x_2 + 4x_3 - x_4 &= 12 \end{aligned}$$

---

[4] Software packages such as Maple often will have several ways of doing things. This is the case for solving systems of linear equations. One variant is to enter b as a matrix with one column by typing and entering

```
b:=matrix([[0],[-2],[7]]);
```

When we then type and enter

```
linsolve(A,b);
```

our solution is given in column form. Another way is to use Maple's *solve* command for solving equations and systems of equations. (With this approach it is not necessary to load Maple's linear algebra package.) To do it this way for the system in Example 1, we type and enter

```
solve({x-y+z=0,2*x-3*y-4*z=-2,-2*x-y+z=7},{x,y,z});
```

10. $$\begin{aligned} x_1 - x_2 + x_3 + x_4 - x_5 &= 0 \\ 2x_1 - x_2 + 2x_3 - x_4 + 3x_5 &= 0 \\ 2x_1 - x_2 - 2x_4 + x_5 &= 0 \\ x_1 + x_2 - x_3 - x_4 + 2x_5 &= 0 \\ 2x_1 + 4x_3 + x_4 + 3x_5 &= 0 \end{aligned}$$

11. $$\begin{aligned} x + 2y + z &= -2 \\ 2x + 2y - 2z &= 3 \end{aligned}$$

12. $$\begin{aligned} 2x - 4y + 6z &= 2 \\ -3x + 6y - 9z &= 3 \end{aligned}$$

13. $$\begin{aligned} x - 2y &= 2 \\ x + 8y &= -4 \\ 2x + y &= 1 \end{aligned}$$

14. $$\begin{aligned} 2x + 3y &= 5 \\ 2x + y &= 2 \\ x - 2y &= 1 \end{aligned}$$

15. $$\begin{aligned} 2x_1 - x_2 - x_3 + x_4 + x_5 &= 0 \\ x_1 - x_2 + x_3 + 2x_4 - 3x_5 &= 0 \\ 3x_1 - 2x_2 - x_3 - x_4 + 2x_5 &= 0 \end{aligned}$$

16. $$\begin{aligned} x_1 - 3x_2 + x_3 - x_4 - x_5 &= 1 \\ 2x_1 + x_2 - x_3 + 2x_4 + x_5 &= 2 \\ -x_1 + 3x_2 - x_3 - 2x_4 - x_5 &= 3 \\ 2x_1 + x_2 - x_3 - x_4 - x_5 &= 6 \end{aligned}$$

Determine conditions on a, b, and c so that the systems of equations in Exercises 17 and 18 have solutions.

17. $$\begin{aligned} 2x - y + 3z &= a \\ x - 3y + 2z &= b \\ x + 2y + z &= c \end{aligned}$$

18. $$\begin{aligned} x + 2y - z &= a \\ x + y - 2z &= b \\ 2x + y - 3z &= c \end{aligned}$$

Determine conditions on a, b, c, and d so that the systems of equations in Exercises 19 and 20 have solutions.

19. $$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= a \\ x_1 - x_2 - x_3 + x_4 &= b \\ x_1 + x_2 + x_3 + x_4 &= c \\ x_1 - x_2 + x_3 + x_4 &= d \end{aligned}$$

20. $$\begin{aligned} x_1 - x_2 + x_3 + x_4 &= a \\ x_1 + x_2 - 2x_3 + 3x_4 &= b \\ 3x_1 - 2x_2 + 3x_3 - 2x_4 &= c \\ 2x_2 - 3x_3 + 2x_4 &= d \end{aligned}$$

Determine if the homogeneous systems of linear equations in Exercises 21–24 have nontrivial solutions. You do not have to solve the systems.

21. $$\begin{aligned} 9x - 2y + 17z &= 0 \\ 13x + 81y - 27z &= 0 \end{aligned}$$

22. $$\begin{aligned} 99x_1 + \pi x_2 - \sqrt{5}x_3 &= 0 \\ 2x_1 + (\sin 1)x_2 + 2x_4 &= 0 \\ 3.38x_1 - ex_3 + (\ln 2)x_4 &= 0 \end{aligned}$$

23. $$\begin{aligned} x - y + z &= 0 \\ 2x + y + 2z &= 0 \\ 3x - 5y + 3z &= 0 \end{aligned}$$

24. $$\begin{aligned} x + y + 2z &= 0 \\ 3x - y - 2z &= 0 \\ 2x - 2y - 4z &= 0 \\ x + 3y + 6z &= 0 \end{aligned}$$

25. We have seen that homogeneous linear systems with fewer equations than variables always have infinitely many solutions. What possibilities can arise for nonhomogeneous linear systems with fewer equations than variables? Explain your answer.

26. Give an example of a system of linear equations with more equations than variables that illustrates each of the following possibilities: Has exactly one solution, has infinitely many solutions, and has no solution.

27. Describe graphically the possible solutions to a system of two linear equations in x, y, and z.

28. Describe graphically the possible solutions to a system of three linear equations in x, y, and z.

Use Maple or another appropriate software package to solve the systems of equations in Exercises 29–32. If you are using Mathematica or MATLAB, see the *Technology Resource Manual* for appropriate commands. (To become more comfortable with the software package you are using, you may wish to practice using it to solve some of the smaller systems in Exercises 1–16 before doing these.)

29. $$\begin{aligned} 7x_1 - 3x_2 + 5x_3 - 8x_4 + 2x_5 &= 13 \\ 12x_1 + 4x_2 - 16x_3 - 9x_4 + 7x_5 &= 21 \\ -22x_1 - 8x_2 + 25x_3 - 16x_4 - 8x_5 &= 47 \\ -52x_1 - 40x_2 + 118x_3 - 37x_4 - 29x_5 &= 62 \\ 46x_1 + 82x_2 - 26x_3 + 44x_4 &= 122 \end{aligned}$$

30. $$\begin{aligned} 69x_1 + 101x_2 + 43x_3 + 30x_4 &= 261 \\ -437x_1 - 735x_2 + 335x_3 + 437x_4 &= -406 \\ 299x_1 + 379x_2 - 631x_3 - 2501x_4 &= -4146 \\ 1863x_1 + 2804x_2 + 62x_3 - 1983x_4 &= 4857 \\ 1748x_1 + 2291x_2 - 461x_3 - 9863x_4 &= 4166 \end{aligned}$$

31.
$$\begin{aligned}
&62x_1 + 82x_2 + 26x_3 - 4x_4 \\
&\qquad + 32x_5 + 34x_6 - 2x_7 - 4x_8 = 0 \\
&93x_1 + 123x_2 + 67x_3 - 36x_4 \\
&\qquad + 106x_5 + 51x_6 + 31x_7 - 188x_8 = 0 \\
&-589x_1 - 779x_2 - 303x_3 + 647x_4 \\
&\qquad - 330x_5 - 323x_6 - 256x_7 - 246x_8 = 0 \\
&403x_1 + 533x_2 + 365x_3 - 2493x_4 \\
&\qquad + 263x_5 + 50x_6 + 981x_7 + 1345x_8 = 0 \\
&2511x_1 + 3321x_2 + 1711x_3 - 2636x_4 \\
&\qquad + 2358x_5 + 1357x_6 + 1457x_7 - 2323x_8 = 0 \\
&2356x_1 + 3116x_2 + 2038x_3 - 6828x_4 \\
&\qquad + 2418x_5 + 1936x_6 + 3596x_7 - 357x_8 = 0
\end{aligned}$$

32.
$$\begin{aligned}
&3.3x_1 + 3.3x_2 + 12.1x_3 + 2.2x_4 \\
&\qquad + 45.1x_5 + 7.7x_6 + 12.1x_7 \\
&\qquad + 35.2x_8 + 1.1x_9 = -3.3 \\
&3x_1 + 3x_2 + 15.8x_3 - 4x_4 \\
&\qquad + 61.4x_5 + 82x_6 + 5x_7 \\
&\qquad + 21.2x_8 + 5.8x_9 = -0.6
\end{aligned}$$

(continued)

$$\begin{aligned}
&-3.3x_1 - 3.3x_2 - 16.1x_3 + 1.8x_4 \\
&\qquad - 61.1x_5 - 9.7x_6 - 10.1x_7 \\
&\qquad - 28.2x_8 - 4.2x_9 = 7.3 \\
&3x_1 + 3x_2 + 15x_3 \\
&\qquad + 56.3x_5 + 8.4x_6 + 13.7x_7 \\
&\qquad + 30.3x_8 + 9.8x_9 = -9.9 \\
&3x_1 + 3x_2 + 11x_3 + 3x_4 \\
&\qquad + 37x_5 + 19.5x_6 + 14x_7 \\
&\qquad + 30.5x_8 - 7.5x_9 = -17 \\
&-3x_1 - 3x_2 - 11x_3 - 3x_4 \\
&\qquad - 41.1x_5 - 3.8x_6 - 5.9x_7 \\
&\qquad - 34.1x_8 + 16.4x_9 = 38.3 \\
&-2.2x_4 + 5.2x_5 - 4.2x_6 \\
&\qquad - 11.6x_7 - 1.4x_8 + 31.2x_9 = 48.2 \\
&4.2x_1 + 4.2x_2 + 19.4x_3 - 3.2x_4 \\
&\qquad + 76.4x_5 - 0.2x_6 + 3.4x_7 \\
&\qquad + 35.8x_8 - 9.6x_9 = -23.2
\end{aligned}$$

1.2 Matrices and Matrix Operations

In the previous section we introduced augmented matrices for systems of linear equations as a convenient way of representing these systems. This is one of many uses of matrices. In this section we will look at matrices from a general point of view.

We should be explicit about exactly what a matrix is, so let us begin with a definition. A **matrix** is a rectangular array of objects called the **entries** of the matrix. (For us, the objects will be numbers, but they do not have to be. For example, we could have matrices whose entries are automobiles or members of a marching band.) We write matrices down by enclosing their entries within brackets (some use parentheses instead) and, if we wish to give a matrix a name, we will do so by using capital letters such as A, B, or C. Here are some examples of matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} -7 & 4 & 4 & 0 & 3 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 \\ -1 \\ 1/2 \\ 4 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & -2 & \pi & 8 \\ -1 & 12 & 3/8 & \ln 2 \\ 1 & 1 & 1 & -1 \\ \sqrt{2} & -7 & 0.9 & -391/629 \end{bmatrix}$$