31. $62x_1 + 82x_2 + 26x_3 - 4x_4 + 32x_5 + 34x_6 - 2x_7 - 4x_8 = 0$

$93x_1 + 123x_2 + 67x_3 - 36x_4 + 106x_5 + 51x_6 + 31x_7 - 188x_8 = 0$

$-589x_1 - 779x_2 - 303x_3 + 647x_4 - 330x_5 - 323x_6 - 256x_7 - 246x_8 = 0$

$403x_1 + 533x_2 + 365x_3 - 2493x_4 + 263x_5 + 50x_6 + 981x_7 + 1345x_8 = 0$

$2511x_1 + 3321x_2 + 1711x_3 - 2636x_4 + 2358x_5 + 1357x_6 + 1457x_7 - 2323x_8 = 0$

$2356x_1 + 3116x_2 + 2038x_3 - 6828x_4 + 2418x_5 + 1936x_6 + 3596x_7 - 357x_8 = 0$

32. $3.3x_1 + 3.3x_2 + 12.1x_3 + 2.2x_4 + 45.1x_5 + 7.7x_6 + 12.1x_7 + 35.2x_8 + 1.1x_9 = -3.3$

$3x_1 + 3x_2 + 15.8x_3 - 4x_4 + 61.4x_5 + 82x_6 + 5x_7 + 21.2x_8 + 5.8x_9 = -0.6$

(*continued*)

$-3.3x_1 - 3.3x_2 - 16.1x_3 + 1.8x_4 - 61.1x_5 - 9.7x_6 - 10.1x_7 - 28.2x_8 - 4.2x_9 = 7.3$

$3x_1 + 3x_2 + 15x_3 + 56.3x_5 + 8.4x_6 + 13.7x_7 + 30.3x_8 + 9.8x_9 = -9.9$

$3x_1 + 3x_2 + 11x_3 + 3x_4 + 37x_5 + 19.5x_6 + 14x_7 + 30.5x_8 - 7.5x_9 = -17$

$-3x_1 - 3x_2 - 11x_3 - 3x_4 - 41.1x_5 - 3.8x_6 - 5.9x_7 - 34.1x_8 + 16.4x_9 = 38.3$

$-2.2x_4 + 5.2x_5 - 4.2x_6 - 11.6x_7 - 1.4x_8 + 31.2x_9 = 48.2$

$4.2x_1 + 4.2x_2 + 19.4x_3 - 3.2x_4 + 76.4x_5 - 0.2x_6 + 3.4x_7 + 35.8x_8 - 9.6x_9 = -23.2$

1.2 Matrices and Matrix Operations

In the previous section we introduced augmented matrices for systems of linear equations as a convenient way of representing these systems. This is one of many uses of matrices. In this section we will look at matrices from a general point of view.

We should be explicit about exactly what a matrix is, so let us begin with a definition. A **matrix** is a rectangular array of objects called the **entries** of the matrix. (For us, the objects will be numbers, but they do not have to be. For example, we could have matrices whose entries are automobiles or members of a marching band.) We write matrices down by enclosing their entries within brackets (some use parentheses instead) and, if we wish to give a matrix a name, we will do so by using capital letters such as A, B, or C. Here are some examples of matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} -7 & 4 & 4 & 0 & 3 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 \\ -1 \\ 1/2 \\ 4 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & -2 & \pi & 8 \\ -1 & 12 & 3/8 & \ln 2 \\ 1 & 1 & 1 & -1 \\ \sqrt{2} & -7 & 0.9 & -391/629 \end{bmatrix}$$

Augmented matrices of systems of linear equations have these forms if we delete the dashed line. In fact, the dashed line is included merely as a convenience to help distinguish the left- and right-hand sides of the equations. If a matrix has m rows (which go across) and n columns (which go up and down), we say the **size** (or **dimensions**) of the matrix is (or are) $m \times n$ (read "m by n"). Thus, for the matrices just given, A is a 2×3 matrix, B is a 1×5 matrix, C is a 4×1 matrix, and D is a 4×4 matrix. A matrix such as B that has one row is called a **row matrix** or **row vector**; a matrix such as C that has one column is called a **column matrix** or **column vector**. Matrices that have the same number of rows as columns (that is, $n \times n$ matrices) are called **square matrices**. The matrix D is an example of a square matrix.

As you would expect, we consider two matrices A and B to be equal, written $A = B$, if they have the same size and entries. For example,

$$\begin{bmatrix} -1 & 2 \\ 1 & 12 \end{bmatrix} = \begin{bmatrix} -1 & 8/4 \\ 2-1 & 3 \cdot 4 \end{bmatrix}$$

while

$$\begin{bmatrix} -1 & 2 \\ 1 & 12 \end{bmatrix} \neq \begin{bmatrix} 5 & 2 \\ 1 & 12 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

The general form of an $m \times n$ matrix A is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}. \tag{1}$$

Notice that in this notation the first subscript i of an entry a_{ij} is the row in which the entry appears and the second subscript j is the column in which it appears. To save writing, we shall often indicate a matrix such as this by simply writing

$$A = [a_{ij}].$$

If we wish to single out the ij-entry of a matrix A, we will write

$$\text{ent}_{ij}(A).$$

For instance, if B is the matrix

$$B = \begin{bmatrix} -1 & 2 & 1 \\ 5 & 4 & -9 \\ 3 & -4 & 7 \end{bmatrix},$$

then

$$\text{ent}_{23}(B) = -9.$$

If $A = [a_{ij}]$ is an $n \times n$ matrix, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the **diagonal entries** of A. The matrix B has diagonal entries $-1, 4, 7$.

We will use the symbol $\mathbb{R}$ to denote the set of real numbers. The set of $m \times n$ matrices with entries from $\mathbb{R}$ will be denoted

$$\boxed{M_{m\times n}(\mathbb{R}).}$$

Thus, for example, in set notation

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mid a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\}.$$ [5]

You have encountered two-dimensional vectors in two-dimensional space (which we will denote by $\mathbb{R}^2$) and three-dimensional vectors in three-dimensional space (which we will denote by $\mathbb{R}^3$) in previous courses. One standard notation for indicating such vectors is to use ordered pairs (a, b) for two-dimensional vectors and ordered triples (a, b, c) for three-dimensional vectors. Notice that these ordered pairs and triples are in fact row matrices or row vectors. However, we will be notationally better off if we use column matrices for two- and three-dimensional vectors. We also will identify the set of two-dimensional vectors with $\mathbb{R}^2$ and the set of three-dimensional vectors with $\mathbb{R}^3$; in other words,

$$\mathbb{R}^2 = M_{2\times 1}(\mathbb{R}) = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

and

$$\mathbb{R}^3 = M_{3\times 1}(\mathbb{R}) = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

in this book. More generally, the set of $n \times 1$ column matrices $M_{n\times 1}(\mathbb{R})$ will be denoted $\mathbb{R}^n$ and we will refer to the elements of $\mathbb{R}^n$ as **vectors in $\mathbb{R}^n$** or **n-dimensional vectors**.

We next turn our attention to the "arithmetic" of matrices beginning with the operations of **addition** and a multiplication by numbers called **scalar multiplication**.[6] If A and B are matrices of the same size, we add A and B by adding their corresponding

---

5 In set notation, the vertical bar, |, denotes "such that" (some use a colon, :, instead of a vertical bar) and the symbol $\in$ denotes "element of" (or "member of"). One way of reading

$$\left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mid a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R} \right\}$$

is as "the set of matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

such that $a_{11}, a_{12}, a_{21}, a_{22}$ are elements of the set of real numbers."

6 These two operations are extensions of the ones you already know for vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ to matrices in general.

entries; that is, if

$$A = [a_{ij}] \quad \text{and} \quad B = [b_{ij}]$$

are matrices in $M_{m \times n}(\mathbb{R})$, the sum of A and B is the $m \times n$ matrix

$$\boxed{A + B = [a_{ij} + b_{ij}].}$$

For instance, if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 8 & 9 \\ 10 & 11 \\ 12 & 13 \end{bmatrix},$$

then

$$A + B = \begin{bmatrix} 1+8 & 2+9 \\ 3+10 & 4+11 \\ 5+12 & 6+13 \end{bmatrix} = \begin{bmatrix} 9 & 11 \\ 13 & 15 \\ 17 & 19 \end{bmatrix}.$$

Note that we have only defined sums of matrices of the same size. The sum of matrices of different sizes is undefined. For example, the sum

$$\begin{bmatrix} -2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & -2 \end{bmatrix}$$

is undefined. If c is a real number (which we call a **scalar** in this setting) and $A = [a_{ij}]$ is an $m \times n$ matrix, the **scalar product** cA is the $m \times n$ matrix obtained by multiplying c times each entry of A:

$$\boxed{cA = c[a_{ij}] = [ca_{ij}].}$$

For example,

$$5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5\cdot 1 & 5\cdot 2 \\ 5\cdot 3 & 5\cdot 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}.$$

The following theorem lists some elementary properties involving addition and scalar multiplication of matrices.

THEOREM 1.2 If A, B, and C are matrices of the same size and if c and d are scalars, then:

1. $A + B = B + A$ (commutative law of addition).
2. $A + (B + C) = (A + B) + C$ (associative law of addition).
3. $c(dA) = (cd)A$.
4. $c(A + B) = cA + cB$.
5. $(c + d)A = cA + dA$.

Proof We prove these equalities by showing that the matrices on each side have the same entries. Let us prove parts (1) and (4) here. The proofs of the remaining parts will be left as exercises (Exercise 24). For notational purposes, we set

$$A = [a_{ij}] \quad \text{and} \quad B = [b_{ij}].$$

Part (1) follows since

$$\text{ent}_{ij}(A + B) = a_{ij} + b_{ij} = b_{ij} + a_{ij} = \text{ent}_{ij}(B + A).$$

To obtain part (4),

$$\text{ent}_{ij}(c(A + B)) = c(a_{ij} + b_{ij}) = ca_{ij} + cb_{ij} = \text{ent}_{ij}(cA + cB).$$ ●

One special type of matrix is the set of **zero matrices**. The $m \times n$ zero matrix, denoted $O_{m\times n}$, is the $m \times n$ matrix that has all of its entries zero. For example,

$$O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad O_{4\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that if A is an $m \times n$ matrix, then:

1. $A + O_{m\times n} = A$.
2. $0 \cdot A = O_{m\times n}$.

We often will indicate a zero matrix by simply writing $\boldsymbol{O}$. (To avoid confusion with the number zero, we put this in boldface print in this book.) For instance, we might write the first property as $A + \boldsymbol{O} = A$. The second property could be written as $0 \cdot A = \boldsymbol{O}$.

The **negative** of a matrix $A = [a_{ij}]$, denoted $-A$, is the matrix whose entries are the negatives of those of A:

$$\boxed{-A = [-a_{ij}].}$$

Notice that

$$-A = (-1)A \quad \text{and} \quad A + (-A) = \boldsymbol{O}.$$

Subtraction of matrices A and B of the same size can be defined in terms of adding the negative of B:

$$\boxed{A - B = A + (-B).}$$

Of course, notice that $A - B$ could also be found by subtracting the entries of B from the corresponding entries of A.

Up to this point, all of the operations we have introduced on matrices should seem relatively natural. Our final operation will be matrix multiplication, which upon first glance may not seem to be the natural way to multiply matrices. However, the manner of multiplying matrices you are about to see is the one that we will need as we use matrix multiplication in our future work.

Here is how we do matrix multiplication: Suppose that $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times l$ matrix. The product of A and B is defined to be the $m \times l$ matrix

$$\boxed{AB = [p_{ij}]}$$

where

$$\boxed{p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.}$$

In other words, for each $1 \le i \le m$ and $1 \le j \le l$ the ij-entry of AB is found by multiplying each entry of row i of A times its corresponding entry of column j of B and then summing these products.

Here is an example illustrating our matrix multiplication procedure.

EXAMPLE 1 Find the product AB for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

Solution The product AB is

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.$$ ●

Once you practice this sum of row entries times column entries a few times, you should find yourself getting the hang of it.[7] Let us do another example of matrix multiplication.

EXAMPLE 2 Find the product CD for

$$C = \begin{bmatrix} -1 & 2 & -3 \\ 0 & -1 & 1 \\ 4 & 2 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 1 & 1 \end{bmatrix}.$$

[7] You might find it convenient to note that the ij-entry of AB is much like the dot product of the vector formed by row i of A with the vector formed by column j of B. We will discuss dot products more fully in Chapter 9.

Solution

$$CD = \begin{bmatrix} -1 & 2 & -3 \\ 0 & -1 & 1 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)\cdot 1 + 2(-3) - 3\cdot 1 & (-1)(-2) + 2\cdot 4 - 3\cdot 1 \\ 0\cdot 1 - 1(-3) + 1\cdot 1 & 0(-2) - 1\cdot 4 + 1\cdot 1 \\ 4\cdot 1 + 2(-3) - 1\cdot 1 & 4(-2) + 2\cdot 4 - 1\cdot 1 \end{bmatrix} = \begin{bmatrix} -10 & 7 \\ 4 & -3 \\ -3 & -1 \end{bmatrix}$$

●

Notice that for the product AB of two matrices A and B to be defined, it is necessary that the number of columns of A be the same as the number of rows of B. If this is not the case, the product is not defined. For instance, the product DC for the matrices in Example 2 is not defined. In particular, CD is not the same as DC. This is an illustration of the fact that *matrix multiplication is not commutative;* that is, AB is not in general the same as BA for matrices A and B. Sometimes these products are not the same because one is defined while the other is not, as the matrices C and D illustrate. But even if both products are defined, it is often the case that they are not the same. If you compute the product BA for the matrices in Example 1, you will find (try it)

$$BA = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix},$$

which is not the same as AB.[8] In the case when $AB = BA$ for two matrices A and B, we say A and B **commute.**

While matrix multiplication is not commutative, some properties that you are used to having for multiplication of numbers do carry over to matrices when the products are defined.

THEOREM 1.3 Provided that the indicated sums and products are defined, the following properties hold where A, B, and C are matrices and d is a scalar.

1. $A(BC) = (AB)C$ (associative law of multiplication)
2. $A(B + C) = AB + AC$ (left-hand distributive law)
3. $(A + B)C = AC + BC$ (right-hand distributive law)
4. $d(AB) = (dA)B = A(dB)$

Proof We will prove the first two parts here and leave proofs of the remaining parts as exercises (Exercise 25). For notational purposes, suppose

$$A = [a_{ij}], \quad B = [b_{ij}], \quad \text{and} \quad C = [c_{ij}].$$

---

[8] This is not the first time you have encountered an example of a noncommutative operation. Composition of functions is noncommutative. The cross product of two three-dimensional vectors is another example of a noncommutative operation.

To prove part (1), we also have to introduce some notation for the sizes of A, B, and C. Suppose A is an $m \times n$ matrix, B is an $n \times l$ matrix, and C is an $l \times h$ matrix. Both $A(BC)$ and $(AB)C$ are $m \times h$ matrices. (Why?) To see that these products are the same, we work out the ij-entry of each. For $A(BC)$, this is

$$\text{ent}_{ij}(A(BC)) = \sum_{k=1}^{n} a_{ik}\text{ent}_{kj}(BC) = \sum_{k=1}^{n} a_{ik}\left(\sum_{q=1}^{l} b_{kq}c_{qj}\right) = \sum_{k=1}^{n}\left(\sum_{q=1}^{l} a_{ik}b_{kq}c_{qj}\right).$$

Carrying out the same steps for $(AB)C$,

$$\text{ent}_{ij}((AB)C) = \sum_{q=1}^{l} \text{ent}_{iq}(AB)c_{qj} = \sum_{q=1}^{l}\left(\sum_{k=1}^{n} a_{ik}b_{kq}\right)c_{qj} = \sum_{q=1}^{l}\left(\sum_{k=1}^{n} a_{ik}b_{kq}c_{qj}\right).$$

Since the summations over k and q are interchangeable, we see that the ij-entries of $A(BC)$ and $(AB)C$ are the same and hence $A(BC) = (AB)C$.

To prove part (2), we again introduce notation for the sizes of our matrices. Suppose A is an $m \times n$ matrix and B and C are $n \times l$ matrices. Both $A(B + C)$ and $AB + AC$ are $m \times l$ matrices. We have

$$\text{ent}_{ij}(A(B + C)) = \sum_{k=1}^{n} a_{ik}(\text{ent}_{kj}(B + C)) = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj})$$
$$= \sum_{k=1}^{n}(a_{ik}b_{kj} + a_{ik}c_{kj})$$

and

$$\text{ent}_{ij}(AB + AC) = \text{ent}_{ij}(AB) + \text{ent}_{ij}(AC) = \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj}$$
$$= \sum_{k=1}^{n}(a_{ik}b_{kj} + a_{ik}c_{kj}).$$

Thus $A(B + C) = AB + AC$ since they have the same entries. ●

If A is a square matrix, we can define positive integer powers of A in the same manner as we do for real numbers; that is,

$$A^1 = A, \quad A^2 = AA, \quad \text{and} \quad A^3 = A^2A = AAA, \ldots.$$

Such powers are not defined, however, if A is not a square matrix. (Why is this the case?)

If A is an $m \times n$ matrix, it is easy to see that

$$O_{l\times m}A = O_{l\times n} \quad \text{and} \quad AO_{n\times l} = O_{m\times l}.$$

Besides zero matrices, another special type of matrices is the set of **identity matrices**. The $n \times n$ identity matrix, denoted I_n, has diagonal entries 1 and all other entries 0. For example,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Identity matrices play the role the number 1 plays for the real numbers with respect to multiplication in the sense that

$$I_m A = A \quad \text{and} \quad A I_n = A$$

for any $m \times n$ matrix A. (Convince yourself of these statements.)

One use (among many more to come) of matrix multiplication arises in connection with systems of linear equations. Given a system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

we will let A denote the coefficient matrix of this system,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

X denote the column of variables,

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

and B denote the column,

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Observe that our system can then be conveniently written as the **matrix equation**

$$\boxed{AX = B.}$$

For instance, the system

$$\begin{aligned} 2x - y + 4z &= 1 \\ x - 7y + z &= 3 \\ -x + 2y + z &= 2 \end{aligned}$$

would be written

$$\begin{bmatrix} 2 & -1 & 4 \\ 1 & -7 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

as a matrix equation. Notice that a homogeneous linear system takes on the form $AX = O$ when written as a matrix equation.

EXERCISES 1.2

In Exercises 1–18, either perform the indicated operations or state that the expression is undefined where A, B, C, D, E, and F are the matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ -3 & -2 \\ 0 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 1 \\ 3 & -1 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & -3 & 5 \\ 2 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -3 & 6 \\ 1 & 0 & 1 \end{bmatrix}.$$

1. $A + B$
2. $D - C$
3. $2B$
4. $-\frac{3}{4}F$
5. $A - 4B$
6. $3D + 2C$
7. CD
8. DC
9. EF
10. FE
11. AE
12. EA
13. $(E + F)A$
14. $B(C + D)$
15. $3AC$
16. $F(-2B)$
17. C^2
18. A^3

Write the systems of equations in Exercises 19 and 20 in the matrix form $AX = B$.

19.
$$\begin{aligned} 2x - y + 4z &= 1 \\ x + y - z &= 4 \\ y + 3z &= 5 \\ x + y &= 2 \end{aligned}$$

20.
$$\begin{aligned} x_1 - 3x_2 + x_3 - 5x_4 &= 2 \\ x_1 + x_2 - x_3 + x_4 &= 1 \\ x_1 - x_2 - x_3 + 6x_4 &= 6 \end{aligned}$$

Write the matrix equations as systems of equations in Exercises 21 and 22.

21. $$\begin{bmatrix} 2 & -2 & 5 & 7 \\ 4 & 5 & -11 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 12 \\ -3 \end{bmatrix}$$

22. $$\begin{bmatrix} 2 & 2 & -11 \\ 0 & -1 & -5 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 51 \\ -33 \\ 1/2 \end{bmatrix}$$

23. Suppose that A and B are $n \times n$ matrices.
    a) Show that $(A + B)^2 = A^2 + AB + BA + B^2$.
    b) Explain why $(A + B)^2$ is not equal to $A^2 + 2AB + B^2$ in general.
24. Prove the following parts of Theorem 1.2.
    a) Part (2)
    b) Part (3)
    c) Part (5)
25. Prove the following parts of Theorem 1.3.
    a) Part (3)
    b) Part (4)
26. Suppose A is an $m \times n$ matrix and B is an $n \times l$ matrix. Further, suppose that A has a row of zeros. Does AB have a row of zeros? Why or why not? Does this also hold if B has a row of zeros? Why or why not?
27. Suppose A is an $m \times n$ matrix and B is an $n \times l$ matrix. Further, suppose that B has a column of zeros. Does AB have a column of zeros? Why or why not? Does this also hold if A has a column of zeros? Why or why not?
28. Give an example of two matrices A and B for which $AB = \mathbf{O}$ with $A \neq \mathbf{O}$ and $B \neq \mathbf{O}$.

29. a) Suppose that A is the row vector

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

and B is an $n \times l$ matrix. View B as the column of row vectors

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix}$$

where $B_1, B_2, \ldots, B_n$ are the rows of B. Show that

$$AB = a_1 B_1 + a_2 B_2 + \cdots + a_n B_n.$$

b) Use the result of part (a) to find AB for

$$A = \begin{bmatrix} -2 & 1 & 6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 1 & 1 \\ 4 & -1 & 2 \end{bmatrix}.$$

30. a) Suppose that B is the column vector

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

and A is an $m \times n$ matrix. View A as the row of column vectors

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \end{bmatrix}$$

where $A_1, A_2, \ldots, A_n$ are the columns of A. Show that

$$AB = b_1 A_1 + b_2 A_2 + \cdots + b_n A_n.$$

b) Use the result of part (a) to find AB for

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 3 & 5 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

31. The **trace** of a square matrix A, denoted $\operatorname{tr}(A)$, is the sum of the diagonal entries of A. Find $\operatorname{tr}(A)$ for

$$A = \begin{bmatrix} 5 & 0 & -4 \\ 2 & -11 & 6 \\ 2 & 10 & 3 \end{bmatrix}.$$

32. Prove the following where A and B are square matrices of the same size and c is a scalar.

a) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$

b) $\operatorname{tr}(cA) = c \operatorname{tr}(A)$

c) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$

The *matrix* command introduced in the previous section is one way of entering matrices on a Maple worksheet. Maple uses the *evalm* command along with +, -, *, &*, and ∧ to find sums, differences, scalar products, matrix products, and matrix powers, respectively. For instance, to find $A - B + 4C + AB - C^3$ where A, B, and C are matrices already entered on a Maple worksheet, we would type and enter

```
evalm(A-B+4*C+A&*B-C^3);
```

at the command prompt. A scalar product cA also may be found with the *scalarmul* command by typing

```
scalarmul(A,c);
```

at the command prompt. Products of two or more matrices can be found by using the *multiply* command. For instance, typing and entering

```
multiply(B,A,C);
```

will give us the product BAC. Use these Maple commands or appropriate commands in another suitable software package (keep in mind that corresponding Mathematica and MATLAB commands can be found in the *Technology Resource Manual*) to find the indicated expression (if possible) where

$$A = \begin{bmatrix} 4 & -2 & 16 & 27 & -11 \\ 9 & 43 & 9 & -8 & -1 \\ 34 & 20 & -3 & 0 & 21 \\ -5 & 4 & 4 & 7 & 41 \\ 0 & 12 & -2 & -2 & 3 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 & 4 \\ 0 & 0 & 0 & 5 & 5 \end{bmatrix}, \text{ and}$$

$$C = \begin{bmatrix} -2 & 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 1 & -1 \\ 3 & -1 & -1 & 1 & 1 \\ -1 & 0 & 2 & 2 & -3 \end{bmatrix}$$

in Exercises 33–40.

33. $A - 2B$

34. $5A + 6C$

35. ABC

36. $CB + C$

37. $(A + B)^2$

38. $4A + CB$

39. $4CA - 5CB - 2C$

40. $B^2 - 4AB + 2A^2$