2.3 Solutions

#6.

$$c_1\begin{bmatrix}0\\4\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\5\\-3\end{bmatrix} + c_3\begin{bmatrix}1\\-3\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Leads to the system

$$\begin{aligned} c_2 + c_3 &= 0 \\ 4c_1 + 5c_2 - 3c_3 &= 0 \\ -c_1 - 3c_2 - c_3 &= 0 \end{aligned}$$

Form the augmented matrix

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 4 & 5 & -3 & 0 \\ -1 & -3 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ the system has a free variable

$\Rightarrow$ infinitely many solutions

$\Rightarrow$ the homogeneous system has a non-trivial solution for c_1, c_2, c_3.

$\Rightarrow$ the vectors are linearly dependent.

#14. $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$ form a basis

for $\mathbb{R}^3$ since

$$\underbrace{\begin{vmatrix} 2 & 1 & 1 \\ -1 & 3 & -4 \\ 0 & -1 & -1 \end{vmatrix}}_{A} = 2\begin{vmatrix} 3 & -4 \\ -1 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}$$

$$= 2(-3-4) = -14 \neq 0$$

$\therefore$ the matrix A is invertible

Hence, the systems

$A\vec{c} = \vec{0}$ and $A\vec{d} = \vec{b}$ (where $\vec{b}$ is arbitrary in $\mathbb{R}^3$)

have unique solutions for $\vec{c}$ & $\vec{d}$

$\vec{c} = A^{-1}\vec{0} = \vec{0} \implies$ the vectors are L.I.

and

$\vec{d} = A^{-1}\vec{b} \implies \vec{b}$ is in the span of the vectors.

$\implies$ they span $\mathbb{R}^3$ since $\vec{b}$ was arbitrary