Math 307 Practice Exam 1, Fall 2022

Name: Solutions

Question	Points	Score
1	16	
2	5	
3	5	
4	5	
5	8	
6	4	
7	6	
8	10	
9	7	
10	8	
Total:	74	

- You have 75 minutes to complete this exam.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work, unless otherwise indicated. You will get almost no credit for solutions that are not fully justified.
- You may not use notes or calculators on this exam.

1. (16 points) True/False questions. No justification necessary.

(a) True False The zero space is a linear subspace of every vector space

(b) True False The only 1×1 matrices that are in row-reduced echelon form are [0] and [1].

(c) True False A matrix of size 3×4 can have 4 leading 1's.

(d) True False For all $n \times n$ matrices A, B and C, $A(B + C) = AB + AC$.

(e) True False If $\det(A) = 0$, then A is invertible.

(f) True False There exist nonzero square matrices A and B such that $(A+B)^2 = A^2 + B^2$.

(g) True False Multiplying a 3×3 matrix A on the left by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has the effect of scaling every entry in row 2 of A by 7.

2.4 will not be on exam!

(h) True False Every basis of $\mathbb{R}^9$ has exactly 9 vectors.

(i) True False For each invertible matrices A and B the matrix AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

(j) True False There is a 2×2 invertible matrix that has 3 entries that are 0.

(k) True False There is a 3×4 matrix A and a 4×3 matrix B such that $AB = BA$.

(l) True False A homogeneous system of linear equations $A\vec{x} = \vec{0}$ is always consistent.

(m) True False If A is not invertible, then A can be row reduced to a matrix with a row of zeros.

(n) True False For any matrix A, the matrix AA^T is symmetric.

(o) True False If $v_1, \ldots, v_n$ are linearly independent vectors in $\mathbb{R}^n$, then they form a basis for $\mathbb{R}^n$.

(p) True False For each $n \times n$ matrix A we have $\det(2A) = 2\det(A)$.

2. (5 points) Which of the following matrices are in RREF. No justification is necessary.

Matrix	Is in RREF	Is NOT in RREF
$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	✓	
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	✓	
$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$	✓	
$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$		✓
$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$		✓

3. (5 points) Which of the following are subspaces of the function space $F(\mathbb{R})$. No justification is necessary.

Set	Is a subspace	Is not a subspace
The space $C^\infty(\mathbb{R})$ of all smooth functions.	✓	
The set of functions f in $F(\mathbb{R})$ such that $f(0) = 0$.	✓	
The set of functions f in $F(\mathbb{R})$ such that $f(2) = 2$.		✓
The polynomials P_3.	✓	
The set of all constant functions $f_c(x) = c$.	✓	

4. (5 points) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 5R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -1 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{array}\right] \begin{array}{l} R_1 + R_3 \\ \\ \\ \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & 5 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ 7 & 5 & 1 \end{bmatrix}$$

5. (8 points) Define the matrices

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 5 \end{bmatrix}$$

Compute the following, or state that the expression is undefined.

(a) $A(-C)$

(b) CA

(c) $B^T B$

(d) BB^T

(e) C^3

(a) $A(-C) = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -1 & -2 \\ -1 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ -3 & -4 \\ -2 & -15 \end{bmatrix}$

(b) not defined.

(c) $B^t B = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 1 & 1 & 3 \\ 6 & 3 & 9 \end{bmatrix}$

(d) $BB^t = \begin{bmatrix} 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4+1+9 \end{bmatrix} = \begin{bmatrix} 14 \end{bmatrix}$

e) not defined.

6. (4 points) Solve the linear system of equations

$$
\begin{aligned}
x_1 - 7x_2 + x_5 &= 3 \\
x_3 - x_5 &= 2 \\
x_4 + x_5 &= 1
\end{aligned}
$$

Form the augmented matrix

$$
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\left[\begin{array}{ccccc|c}
1 & -7 & 0 & 0 & 1 & 3 \\
0 & 0 & 1 & 0 & -1 & 2 \\
0 & 0 & 0 & 1 & 1 & 1
\end{array}\right]
\end{array}
$$

Already in RREF. Let x_2 and x_5 be free.

and

$$x_1 = 7x_2 - x_5 + 3$$

$$x_3 = x_5 + 2$$

$$x_4 = -x_5 + 1.$$

7. (6 points) Compute the determinant of the matrices:

(a) $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

upper triangular

$\Rightarrow \det(A) = 1.$

product of diagonal entries.

(b) $B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$ *Hint*: do one row operation first.

$R_5 - R_4$

(c) $C = \begin{bmatrix} 3 & 0 & 0 & 2 \\ 8 & 4 & 7 & 0 \\ 1 & 1 & 0 & -1 \\ 2 & 3 & 0 & 0 \end{bmatrix}$

$$\det(B) = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

$= 0.$

(c) Cofactor expansion along column 3.

$$\det(C) = 0 - 7 \cdot \begin{vmatrix} 3 & 0 & 2 \\ 1 & 1 & -1 \\ 2 & 3 & 0 \end{vmatrix} + 0 - 0$$

$$= -7 \left[3 \cdot \begin{vmatrix} 1 & -1 \\ 3 & 0 \end{vmatrix} - 0 + 2 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} \right]$$

$$= -7(3(0+3) + 2(3-2))$$

$$= -7(9+2) = -77.$$

8. (10 points) Let

$$\alpha = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(a) Are the vectors in α linearly dependent or independent?

(b) Is the vector

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

in the span of α.

(c) Do the vectors in α span $\mathbb{R}^4$? Why?

(d) Do the vectors in α form a basis for Span(α)? Why?

(e) Find $v \in \mathbb{R}^4$ if

$$[v]_\alpha = \begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}.$$

(a) Put

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This ~~sys~~ leads to a system of linear eqns in c_1, c_2, c_3. with corrosponding augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ R_1 + R_2 \\ \\ R_4 - 2R_1 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 3 & 0 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ R_2 - 2R_3 \\ \\ R_4 + 2R_3 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ \\ R_4 - 3R_2 \end{array}$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 \leftrightarrow R_3 \\ \\ \\ \end{array}$$

continue →

(more space for problem 8)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow c_1 = c_2 = c_3 = 0 \text{ is the unique solution.}$$

$\Longrightarrow$ The vectors in α are L.I.

(b) Performing the same sequence of Row operations as above to the column vector

$$\begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \begin{matrix} R_1+R_2 \\ \\ R_4-2R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 \\ 0 \\ 2 \\ -2 \end{bmatrix} \begin{matrix} R_1-R_3 \\ R_2-2R_3 \\ R_4+2R_3 \end{matrix} \rightarrow \begin{bmatrix} -1 \\ -4 \\ 2 \\ 2 \end{bmatrix} \begin{matrix} R_1+R_2 \\ \\ R_4-3R_2 \end{matrix}$$

$$\rightarrow \begin{bmatrix} -5 \\ -4 \\ 2 \\ +14 \end{bmatrix} R_2 \leftrightarrow R_3 \rightarrow \begin{bmatrix} -5 \\ 2 \\ -4 \\ 14 \end{bmatrix}$$

$$\therefore \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ -1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 14 \end{array}\right]$$

system has no solution

$\Longrightarrow$ vector not in span.

(c) No. We need at least 4 vectors.

(d) Yes. The vectors in α are L.I. and span span(α).

(e) $$v = 2 \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} + 7 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 7 \\ 3 \end{bmatrix}$$

9. (7 points) Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} s \\ t \end{bmatrix}$$

where the entries in A and $\vec{b}$ are real numbers and the entries in $\vec{x}$ are variables. Suppose A has nonzero determinant. Use Cramer's rule to derive formulas for the solutions to the system $A\vec{x} = \vec{b}$.

$$A_1 = \begin{bmatrix} s & b \\ t & d \end{bmatrix} \Rightarrow \det A_1 = sd - bt$$

$$A_2 = \begin{bmatrix} a & s \\ c & t \end{bmatrix} \Rightarrow \det A_2 = at - cs$$

$$\det A = ad - bc$$

$$\therefore \quad x = \frac{sd - bt}{ad - bc}$$

$$y = \frac{at - cs}{ad - bc}$$

10. (8 points) Let

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 7 & 8 & 3 \\ 4 & 5 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

Compute

(a) $\det(A^T B)$

(b) $\det((B^{-1})^3 A^2)$

(c) $\det(A - 2B)$

$$\det A = -1 \begin{vmatrix} 7 & 3 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} 7 & 8 \\ 4 & 5 \end{vmatrix}$$

$$= -1(14-12) + 2(35-32)$$

$$= -1 \cdot 2 + 2 \cdot 3 = 4$$

$$\det B = 3 \cdot 2 \cdot 1 = 6$$

(a) $\det(A^T B) = \det(A^T) \cdot \det(B) = \det(A) \cdot \det(B)$

$= 4 \cdot 6$

$= 24$

(b) $\det((B^{-1})^3 \cdot A^2) = \det((B^{-1})^3) \cdot \det(A^2)$

$= \det(B^{-1})^3 \det(A)^2$

$= \left(\frac{1}{\det(B)}\right)^3 \cdot \det(A)^2$

$= \left(\frac{1}{6}\right)^3 \cdot 4^2$

$= \frac{2}{27}$

cont. → on back

$$A - 2B = \begin{bmatrix} 0 & 1 & 2 \\ 7 & 8 & 3 \\ 4 & 5 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 8 & 4 & 0 \\ 6 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 1 & 2 \\ -1 & 4 & 3 \\ -2 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow \det(A-2B) = 2\begin{vmatrix} -1 & 4 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} -6 & 1 \\ -2 & 1 \end{vmatrix}$$

$$= 2(-1+8) - 3(-6+2)$$

$$= 14 + 12$$

$$= 26.$$