#2.

$$\left[\begin{array}{ccc|c} 3 & 11 & 19 & -2 \\ 7 & 23 & 39 & 10 \\ -4 & -3 & -2 & 6 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \leftarrow$$

$\Rightarrow 0 = 1$ system has no solution.

#3.

$$\text{(i)} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/12 & -1/3 & 5/12 \\ 0 & 1 & 0 & -5/12 & 2/3 & -1/12 \\ 0 & 0 & 1 & 7/12 & -1/3 & -1/12 \end{array}\right]$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1/12 & -1/3 & 5/12 \\ -5/12 & 2/3 & -1/12 \\ 7/12 & -1/3 & -1/12 \end{bmatrix}$$

$$\text{(ii)} \quad A^T = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\text{(iii)} \quad 3A = \begin{bmatrix} 3 & 6 & 9 \\ 3 & 9 & 6 \\ 9 & 6 & 3 \end{bmatrix}$$

$$\text{(iv)} \quad A^2 = A \cdot A$$

$$= \begin{bmatrix} 12 & 14 & 10 \\ 10 & 15 & 11 \\ 8 & 14 & 14 \end{bmatrix}$$

#4.

$$\det A = \begin{vmatrix} 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \end{vmatrix} \quad R_1 \leftrightarrow R_4$$

$$= (-1)\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 2 & 7 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \end{vmatrix} \quad R_2 \leftrightarrow R_4$$

$$= (-1)^2\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 4 & 5 \end{vmatrix}$$

$$= (-1)^3 \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix}$$

$$= (-1)^3 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = -120$$

$\det B = 1^5 = 1.$

$$\det C = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{vmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \\ R_4 - R_1 \\ R_5 - R_1 \end{array}$$

$$= \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{vmatrix}$$

$$= 1 \cdot 1 \cdot 2 \cdot 3 \cdot 4 = 24$$

(a) $\det(2A) = 2^5 \cdot \det(A) = 32 \cdot (-120) = -3840$

(b) $\det(A^T \cdot B^2 \cdot C^{-1})$

$$= \det(A^T) \cdot \det(B^2) \cdot \det(C^{-1})$$

$$= \det(A) \cdot (\det B)^2 \cdot \det(C)^{-1}$$

$$= (-120) \cdot 1^2 \cdot 1/24$$

$$= -5$$

(c) det(A+B)

$$= \begin{vmatrix} 5 & 6 & 6 & 6 & 6 \\ 4 & 3 & 2 & 1 & 5 \\ 3 & 2 & 4 & 4 & 5 \\ 2 & 3 & 3 & 4 & 5 \\ 1 & 0 & 0 & 4 & 5 \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 6 & 6 & 6 & 6 \\ 3 & 2 & 1 & 5 \\ 2 & 4 & 4 & 5 \\ 3 & 3 & 4 & 5 \end{vmatrix} - 4 \cdot \begin{vmatrix} 5 & 6 & 6 & 6 \\ 4 & 3 & 2 & 5 \\ 3 & 2 & 4 & 5 \\ 2 & 3 & 3 & 5 \end{vmatrix}$$

$$+ 5 \begin{vmatrix} 5 & 6 & 6 & 6 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 4 & 4 \\ 2 & 3 & 3 & 4 \end{vmatrix}$$

Do each 4x4 separately.

$$\begin{vmatrix} 6 & 6 & 6 & 6 \\ 3 & 2 & 1 & 5 \\ 2 & 4 & 4 & 5 \\ 3 & 3 & 4 & 5 \end{vmatrix}$$

$$= 6 \cdot \begin{vmatrix} 2 & 1 & 5 \\ 4 & 4 & 5 \\ 3 & 4 & 5 \end{vmatrix} - 6 \begin{vmatrix} 3 & 1 & 5 \\ 2 & 4 & 5 \\ 3 & 4 & 5 \end{vmatrix} + 6 \begin{vmatrix} 3 & 2 & 5 \\ 2 & 4 & 5 \\ 3 & 3 & 5 \end{vmatrix}$$

$$- 6 \begin{vmatrix} 3 & 2 & 1 \\ 2 & 4 & 4 \\ 3 & 3 & 4 \end{vmatrix} \longrightarrow$$

$$= 6\left[2(20-20) - 1(20-15) + 5(16-12)\right]$$

$$-6\left[3(20-20) - 1(10-15) + 5(8-12)\right]$$

$$+6\left[3(20-15) - 2(10-15) + 5(6-12)\right]$$

$$-6\left[3(16-12) - 2(8-12) + 1(6-12)\right]$$

$$= 6\left[0 - 5 + 20\right] - 6\left[0 + 5 - 20\right]$$

$$+6\left[15 + 10 - 30\right] - 6\left[12 + 8 - 6\right]$$

$$= 6\left(15 + 15 - 5 - 14\right)$$

$= 66$ you get the picture

Do the other two the same way, get

$$\det(A+B) = 258$$

#5.

(a) $W = \{ y : y'' = 4y \}$.

(i) Let $y_1, y_2 \in W$, then

$$(y_1 + y_2)'' = y_1'' + y_2'' = 4y_1 + 4y_2 = 4(y_1 + y_2)$$

$$\Longrightarrow y_1 + y_2 \in W.$$

(ii) Let $y_1 \in W$ and $c \in \mathbb{R}$, then

$$(cy_1)' = c \cdot y_1' = c(4y_1) = 4(cy_1)$$

$$\Longrightarrow cy_1 \in W.$$

(b) $W = \{ z \}$.

(i) Let $y_1, y_2 \in W$, then

$$\begin{aligned}(y_1 + y_2)(x) &= y_1(x) + y_2(x) \\ &= z(x) + z(x) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

$$\Longrightarrow y_1 + y_2 = z$$

so $y_1 + y_2 \in W$.

(ii) Let $y_1 \in W$ and $c \in \mathbb{R}$

$$(cy_1)(x) = c \cdot y_1(x) = c \cdot z(x)$$
$$= c \cdot 0$$
$$= 0$$

$\Rightarrow cy_1 = z$

so $cy_1 \in W$.

(c) $W = \{ y : y(x) = c \text{ for some constant } c \}$

(i) Let $y_1, y_2 \in W$, say

$y_1(x) = c_1$ and $y_2(x) = c_2$

Then

$$(y_1 + y_2)(x) = y_1(x) + y_2(x)$$
$$= c_1 + c_2$$

$\Rightarrow y_1 + y_2$ is a constant function

$\Rightarrow y_1 + y_2 \in W$.

(ii) Let $y_1 \in W$ and $c \in \mathbb{R}$, then

$$(cy_1)(x) = c \cdot y_1(x)$$

$$= c \cdot c_1$$

$\longrightarrow$ cy_1 is a constant function.

$\Longrightarrow$ $cy_1 \in W$.

(d) Let $W = \{ p : p \text{ is a polynomial} \}$

(i) Let $p, q \in W$, say

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

$$q(x) = b_n x^n + \cdots + b_1 x + b_0$$

(the degrees may be different but we can always add 0 without changing the polynomial

$$0x^n + 0x^{n-1} + \cdots + 0x^{N+1} + b_N x^N + \cdots + b_1 x + b_0)$$

Then

$$(p+q)(x) = p(x) + q(x)$$

$$= \left(a_n x^n + \cdots + a_1 x + a_0\right) + \left(b_n x^n + \cdots + b_1 x + b_0\right)$$

$$= (a_n + b_n)x^n + \cdots + (a_1 + b_1)x + (a_0 + b_0)$$

$\Rightarrow$ $p+q$ is a polynomial.

$\Rightarrow$ $p+q \in W$.

(ii) Let $p \in W$ and $c \in \mathbb{R}$, then

$$(cp)(x) = c \cdot p(x)$$

$$= c \cdot \left(a_n x^n + \cdots + a_1 x + a_0\right)$$

$$= (ca_n)x^n + \cdots + (ca_1)x + (ca_0)$$

$\Rightarrow$ cp is a polynomial

$\Rightarrow$ $cp \in W$.

(e) $W = \{f : f \text{ is cts on } \mathbb{R}\}$.

(i) Let $f, g \in W$, then from calc. I we know

$f+g$ is cts.

Review: If $a \in \mathbb{R}$

$$\lim_{x \to a} (f(x)+g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a)$$

(ii) Also from calc I, cf is cts.

(f) Let $W = \{f : f \text{ is smooth}\}$.

From calc I. If f, g are smooth, then $f+g$ is smooth. Same for cf.

(g) $W = \{f : f(2) = 1\}$

Is not a subspace since for $f \in W$,

$$(3f)(2) = 3 \cdot f(2) = 3 \cdot 1 = 3 \neq 1.$$

so $3f \notin W$.

(h) $W = \{ f : f(2) = 0 \}$

(i) Let $f, g \in W$, then

$$(f+g)(2) = f(2) + g(2)$$
$$= 0 + 0$$
$$= 0$$

$\Rightarrow f+g \in W$

(ii) Let $f \in W$ and $c \in \mathbb{R}$, then

$$(cf)(2) = c \cdot f(2) = c \cdot 0 = 0$$

$\Rightarrow cf \in W.$

#7.

(a) $$c_1\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + c_3\begin{bmatrix}1\\3\\6\\9\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c}1&1&1&0\\1&2&3&0\\1&3&6&0\\1&4&9&0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right]$$

$\Rightarrow c_1 = c_2 = c_3 = 0$

$\Rightarrow$ the vectors are L.I.

(b) Are there scalars c_1, c_2, c_3 s.t.

$$c_1\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + c_3\begin{bmatrix}1\\3\\6\\9\end{bmatrix} = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c}1&1&1&1\\1&2&3&0\\1&3&6&1\\1&4&9&0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{array}\right]$$

System is inconsistent

$\Rightarrow$ vector is not in span.

(c) Not a basis since they do not span $\mathbb{R}^4$ (see (b))

#7

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix} \to \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} d & f & d & f & d \end{matrix} \end{array}$$

x_2, x_4 are free

$x_1 = x_2 - 2x_4$

$x_3 = -x_4$

$x_5 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in NS(A) \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_4 \\ x_2 \\ -x_4 \\ x_4 \\ 0 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ is a basis of $NS(A)$.

$$[0\ \ -1\ \ 0\ \ 2\ \ 0],\ [0\ \ 0\ \ 1\ \ 1\ \ 0],\ [0\ \ 0\ \ 0\ \ 0\ \ 1]$$

is a basis of $RS(A)$.

$$A^t = \begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 1 & -1 & -2 \\ -1 & 0 & -2 & -1 \\ 1 & -2 & 0 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 \\ 0 \\ 0 \\ 11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \text{ is a basis of } CS(A)$$

#8

(a) $T(A) = A + I_3$ is not a linear transformation.

Since $T(2I_3) = 2I_3 + I_3 = 3I_3$

$\neq 4I_3$

$= 2(I_3 + I_3)$

$= 2T(I_3)$

(b) $T(A) = PAQ$ is linear

$$
\begin{aligned}
T(cA + dB) &= P(cA + dB)Q \\
&= P(cA)Q + P(dB)Q \\
&= c(PAQ) + d(PBQ) \\
&= c \cdot T(A) + d \cdot T(B)
\end{aligned}
$$

(c) $T(f(x)) = f(x) + x$ is not linear

$$
\begin{aligned}
\text{Since } T(2x) &= 2x + x = 3x \\
&\neq 4x \\
&= 2(x + x) \\
&= 2 \cdot T(x)
\end{aligned}
$$

(d) $T(f(x)) = f''(x) - 2f'(x) + f(x)$ is linear

$T(cf + dg)$

$= (cf+dg)'' - 2(cf+dg)' + (cf + dg)$

$= (cf)'' + (dg)'' - 2(cf)' - 2(dg)' + cf + dg$

$= cf'' + dg'' - c \cdot 2f' - d \cdot 2g' + cf + dg$

$= (cf'' - c \cdot 2f' + cf) + (dg'' - d2g' + dg)$

$= c(f'' - 2f' + f) + d(g'' - 2g' + g)$

$= c.T(f) + d.T(g)$

\#9. $L = (D-1)(D^2+1)$

roots: $1, \pm i$

using 1 ~~&~~ $+i$ ($-i$ works also) we get

$y_1 = e^{1 \cdot x}, \quad y_2 = \cos x, \quad y_3 = \sin x$

Since $\dim(\ker L) = 3$ it suffices to show y_1, y_2 & y_3 are L.I.

$$W(y_1, y_2, y_3) = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix}$$

$$= e^x(\sin^2 x + \cos^2 x) - \cos x(-e^x \sin x - e^x \cos x)$$
$$+ \sin x(-e^x \cos x + e^x \sin x)$$

$$= e^x \cdot 1 + e^x \sin x \cos x + e^x \cos^2 x + e^x \sin^2 x - e^x \sin x \cos x$$

$$= 2e^x$$

$$\neq 0$$

$\therefore$ L.I.

#10. $T(v_1) = 2w_1 - w_2$

$T(v_2) = -w_1 + 2w_2$

(a) $[T]_\alpha^\beta = \big[\, [Tv_1]_\beta \;\; [Tv_2]_\beta \,\big]$

$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

(b) $[Tv]_\beta = [T]_\alpha^\beta [v]_\alpha$

$= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix}$

$= \begin{bmatrix} -7 \\ 11 \end{bmatrix}$

(c) $[T]_\alpha^\alpha = P[T]_\alpha^\beta = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

λ is an eigenvalue of $T \iff \lambda$ is an eigenvalue of $[T]_\alpha^\alpha$

$\therefore$ the eigenvalues of T are $1, 2$.

(d) yes, T is diagonalizabl sinc $[T]_\alpha^\alpha$ is.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = [T]_\alpha^\alpha = I^{-1}[T]_\alpha^\alpha \cdot I$$

#11. $H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$$\det(\lambda I - H) = \begin{vmatrix} \lambda-1 & 0 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & 0 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1) \cdot \begin{vmatrix} \lambda-1 & -1 \\ -1 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)\left[(\lambda-1)^2 - 1\right]$$

$$= (\lambda-1)(\lambda^2 - 2\lambda)$$

$$= \lambda(\lambda-1)(\lambda-2)$$

$\lambda = 0, 1, 2$

$\lambda = 0$: $\left[\begin{array}{ccc|c} -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

z is free, $x = -z$, $y = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_0 \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is a basis of E_0.

$$\lambda = 1: \quad \left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

y free, $x = 0$, $z = 0$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_1 \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} y$$

So $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is a basis for E_1

~~Since $\dim(E_0) + \dim(E_1) = 1 + 1 \neq 3$~~

~~A is not diagonalizable~~

cont. →

$\lambda = 2$: $\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ -1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

z free, $x = z$, $y = 2z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_2 \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 2z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a basis for E_2.

Since

$\dim(E_0) + \dim(E_1) + \dim(E_2) = 3$

$\therefore$ H is diagonalizable

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det(\lambda I - T) = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda - 1 & 0 \\ 0 & -1 & \lambda \end{vmatrix}$$

$$= (\lambda - 1)\left[(\lambda - 1)\lambda + 0 \right]$$

$$= (\lambda - 1)^2 \cdot \lambda$$

$\lambda = 0, 1.$

$$\lambda = 0: \quad \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

z free, $x = -z$, $y = 0$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_0 \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is a basis of E_0.

$\lambda = 1$: $\left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

x free, $y = z = 0$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_1 \iff \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis for E_1

Since $\dim(E_0) + \dim(E_1) = 2 \neq 3$

T is not diagonalizable.

#13. Solve $Y' = AY$ (A not diagonalizable)

(1) Find the Jordan canonical form of A

$$J = P^{-1}AP$$

(2) Solve $Y' = JY$ using "back substitution" to get the general solution Z

(3) Apply P to Z, PZ, to get the general solution to $Y' = AY$.

#12. $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 0 & 0 \\ -1 & \lambda-2 & -1 \\ 0 & 0 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)\left[(\lambda-2)(\lambda-1) - 0 \right)$$

$$= (\lambda-1)^2(\lambda-2)$$

$\lambda = 1, 2.$

$\lambda = 1$: $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

y, z free $x = -y - z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} = -y\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

basis for E_1

$$\lambda = 2: \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

y free, $x = z = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

basis of E_2.

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The solutions to $Y' = DY$ is

$$Z = \begin{bmatrix} c_1 e^x \\ c_2 e^x \\ c_3 e^{2x} \end{bmatrix}$$

$$\Longrightarrow PZ = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 e^x \\ c_2 e^x \\ c_3 e^{2x} \end{bmatrix} = \begin{bmatrix} c_1 e^x - c_2 e^x \\ c_3 e^{2x} \\ c_2 e^x \end{bmatrix}$$

$$\Rightarrow M = \begin{bmatrix} e^x & -e^x & 0 \\ 0 & 0 & e^{2x} \\ 0 & e^x & 0 \end{bmatrix}$$ is the matrix of fund. solutions.

$$[M \mid I] = \left[\begin{array}{ccc|ccc} e^x & -e^x & 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2x} & 0 & 1 & 0 \\ 0 & e^x & 0 & 0 & 0 & 1 \end{array}\right] R_1 + R_3$$

$$\rightarrow \left[\begin{array}{ccc|ccc} e^x & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & e^{2x} & 0 & 1 & 0 \\ 0 & e^x & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} e^x & 0 & 0 & 1 & 0 & 1 \\ 0 & e^x & 0 & 0 & 0 & 1 \\ 0 & 0 & e^{2x} & 0 & 1 & 0 \end{array}\right]$$

$$M^{-1} = \begin{bmatrix} e^{-x} & 0 & e^{-x} \\ 0 & 0 & e^{-x} \\ 0 & e^{-2x} & 0 \end{bmatrix}$$

$$\Rightarrow M^{-1}G(x) = \begin{bmatrix} e^{-x} & 0 & e^{-x} \\ 0 & 0 & e^{-x} \\ 0 & e^{-2x} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-x} + xe^{-x} \\ 0 \\ xe^{-2x} \end{bmatrix}$$

$$\int e^{-x} + xe^{-x}\,dx = \int (1+x)e^{-x}\,dx$$

$$\begin{array}{ccc} 1+x & + & e^{-x} \\ 1 & - & -e^{-x} \\ 0 & & e^{-x} \end{array}$$

$$= -(1+x)e^{-x} - e^{-x}$$

$$= e^{-x}(-1+x-1)$$

$$= e^{-x}(x-2)$$

$$\int xe^{-2x} = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}$$

$$\begin{array}{ccc} x & & e^{-2x} \\ 1 & + & \frac{e^{-2x}}{-2} \\ 0 & - & \frac{e^{-2x}}{4} \end{array}$$

$$= e^{-2x}\left(-\frac{x}{2} - \frac{1}{4}\right)$$

$$= -\frac{e^{-2x}}{4}(2x+1)$$

$$\Rightarrow \int M^{-1}G(x)\,dx = \begin{bmatrix} e^{-x}(x-2) \\ 0 \\ -\frac{e^{-2x}}{4}(2x+1) \end{bmatrix}$$

$$M \cdot \int M^{-1} G(x)\, dx$$

$$= \begin{bmatrix} e^{x} & -e^{+x} & 0 \\ 0 & 0 & e^{2x} \\ 0 & e^{x} & 0 \end{bmatrix} \begin{bmatrix} e^{-x}(x-2) \\ 0 \\ -\frac{e^{-2x}}{4}(2x+1) \end{bmatrix}$$

$$= \begin{bmatrix} (x-2) \\ -\frac{1}{4}(2x+1) \\ 0 \end{bmatrix}$$

$\therefore$ the general solution to $Y' = AY + G(x)$ is

$$Y = \begin{bmatrix} c_1 e^{x} - c_2 e^{x} \\ c_3 e^{2x} \\ c_2 e^{x} \end{bmatrix} + \begin{bmatrix} x-2 \\ -\frac{1}{4}(2x+1) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{x} - c_2 e^{x} + x - 2 \\ c_3 e^{2x} - \frac{1}{4}(2x+1) \\ c_2 e^{x} \end{bmatrix}$$

#14.

y_1 - salt in tank 1

y_2 - salt in tank 2.

$$y_1' = \text{Rate in} - \text{Rate out}$$

$$= 3\frac{L}{min} \cdot \frac{y_2}{10}\frac{g}{L} - 6\frac{L}{min} \cdot \frac{y_1}{20}\frac{g}{L}$$

$$= \frac{3}{10} y_2 \frac{g}{min} - \frac{3}{10} y_1 \frac{g}{min}$$

$$y_2' = \text{Rate in} - \text{Rate out.}$$

$$= 4\frac{L}{min} \cdot \frac{y_1}{20}\frac{g}{L} - 3\frac{L}{min} \cdot \frac{y_2}{10}\frac{g}{L}$$

$$= \frac{1}{5} y_1 \frac{g}{min} - \frac{3}{10} y_2 \frac{g}{min}$$

$$Y' = \begin{bmatrix} -3/10 & 3/10 \\ 1/5 & -3/10 \end{bmatrix} Y$$