Math 307 Practice Final, Fall 2022

Name:

Question	Points	Score
1	0	
2	0	
3	0	
4	0	
5	0	
6	0	
7	0	
8	0	
9	0	
10	0	
11	0	
12	0	
13	0	
14	0	
Total:	0	

- You have 120 minutes to complete this exam.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work, unless otherwise indicated. You will get almost no credit for solutions that are not fully justified.
- You may not use notes or calculators on this exam.
- The back side of each page can be used as scratch work or for longer solutions. Please indicate clearly if your solution extends to the back side of a page.

1. True/False questions. No justification necessary.

(a) True False If a matrix is in row-reduced echelon form, then at least one of its entries must be 1.

(b) True False The system

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

is inconsistent.

(c) True False $(A^2)^{-1} = (A^{-1})^2$ whenever A is invertible.

(d) True False The matrix AA^T is symmetric for any matrix A.

(e) True False For any $n \times n$ matrices A and B we have $\det(AB) = \det(BA)$.

(f) True False If W is a subspace of the vector space V and W is finite dimensional, then V is finite dimensional.

(g) True False The kernel of the derivative operator $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ is finite dimensional.

(h) True False If $v_1, \ldots, v_n$ are linearly independent in an n-dimensional space V, then $v_1, \ldots, v_n$ is a basis for V.

(i) True False If $y_1, y_2, y_3 \in C^\infty(a, b)$ and $w(y_1(x), y_2(x), y_3(x)) \neq 0$ for some $x \in (a, b)$, then y_1, y_2, y_3 are linearly independent.

(j) True False The dimension of the kernel of the operator

$$D^9 + (1 - x)D^7 + \sin x D^5 - x^3 D - 1,$$

over $\mathbb{R}$ is 9.

(k) True False If α is the standard basis of $\mathbb{R}^4$ and $\beta = \{v_1, v_2, v_3, v_4\}$ is a basis of $\mathbb{R}^4$, then the change of basis matrix from β to α is the matrix $[v_1 \ v_2 \ v_3 \ v_4]$.

(l) True False Different matrices can have the same characteristic polynomial.

(m) True False Let A is a 5×5 matrix with eigenvalues $\sqrt{2}$ and 1. If the dimension of $E_{\sqrt{2}}$ is 3 and the dimension of E_1 is 2, then A is diagonalizable.

(n) True False The dimension of the space of solutions to the homogeneous system of equations

$$Y' = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} Y$$

is 4.

(o) True False If A is a matrix with $\dim(CS(A)) = 4$ and $\dim(NS(A)) = 12$, then A has 16 rows.

(p) True False The homogeneous system

$$Y' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} Y$$

has matrix of fundamental solutions

$$M = \begin{bmatrix} e^{x} & 0 & 0 \\ 0 & e^{2x} & 0 \\ 0 & 0 & e^{3x} \end{bmatrix}$$

(q) True False The general solution to the nonhomogeneous system of n first-order linear differential equations

$$Y' = A(x)Y + G(x)$$

is

$$Y = M\vec{c} + M \int M^{-1}G(x)dx,$$

where M is the matrix of fundamental solutions to the homogeneous system and $\vec{c}$ is an arbitrary vector in $\mathbb{R}^n$.

(r) True False If $r = 2i$ is a complex root with multiplicty 3 of the characteristic polynomial of the n-th order linear differential equation $Ly = 0$, then

$$\sin(2x), \ \cos(2x), \ x\sin(2x), \ x\cos(2x), \ x^2\sin(2x), \ x^2\cos(2x)$$

are solutions to $Ly = 0$.

2. Solve the system of equations.

$$\begin{aligned} 3x + 11y + 19z &= -2 \\ 7x + 23y + 39z &= 10 \\ -4x - 3y - 2z &= 6 \end{aligned}$$

3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix},$$

Find A^{-1}, A^T, $3A$, and A^2.

4. Let

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 4 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 4 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}$$

Compute the following *Hint:* use properties of determinants.

(a) $\det(2A)$

(b) $\det(A^T B^2 C^{-1})$

(c) $\det(A + B)$

5. Determine which of the following sets are subspaces of $F(\mathbb{R})$. Justify your answer.
   (a) The set of functions that are solutions to the differential equation $y'' = 4y$.
   (b) The set containing just the zero function $z : \mathbb{R} \to \mathbb{R}$ where $z(x) = 0$.
   (c) The constant functions.
   (d) The polynomials.
   (e) The continuous functions.
   (f) The smooth functions.
   (g) All functions whose graph passes through the point $(2, 1)$.
   (h) All functions whose graph passes through the point $(2, 0)$.

6. Let
$$\alpha = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \\ 9 \end{bmatrix} \right\},$$
   (a) Are the vectors in α linearly independent? Justify.
   (b) Is the vector $[1\ 0\ 1\ 0]^T$ in the span of α? Justify.
   (c) Does α form a basis of $\mathbb{R}^4$? Justify.

7. Let
$$A = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & -2 & 2 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix}$$
   Find bases for $NS(A)$, $CS(A)$, and $RS(A)$.

8. Which of the following are linear transformations? Justify.
   (a) $T : M_3(\mathbb{R}) \to M_3(\mathbb{R})$; $T(A) = A + I_3$.
   (b) $T : M_3(\mathbb{R}) \to M_3(\mathbb{R})$; $T(A) = PAQ$ where P and Q are fixed matrices in $M_3(\mathbb{R})$.
   (c) $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$; $T(f(x)) = f(x) + x$.
   (d) $T : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$; $T(f(x)) = f''(x) - 2f'(x) + f(x)$.

9. Find a basis for the Kernel of the differential operator
$$L = (D - 1)(D^2 + 1),$$
   over $\mathbb{R}$. Justify your answer.

10. Let $\alpha = \{v_1, v_2\}$ and $\beta = \{w_1, w_2\}$ be a basis for the vector space V. Suppose the change of basis matrix from α to β is

$$P = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}$$

Let $T : V \to V$ be the linear transformation such that

$$T(v_1) = 2w_1 - w_2$$
$$T(v_2) = -w_1 + 2w_2.$$

(a) Find $[T]_\alpha^\beta$,

(b) Find $[Tv]_\beta$ if $v = -v_1 + 5v_2$.

(c) Find the eigenvalues λ of T and bases for the eigenspaces V_λ.

(d) Is T diagonalizable?

11. Determine if the following matrices are diagonalizable. If a matrix is, give a diagonal matrix D and a matrix P such that $D = P^{-1}AP$ Justify your answer.

$$H = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

12. Solve the nonhomogeneous system

$$Y' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} Y + \begin{bmatrix} 1 \\ x \\ 0 \end{bmatrix}$$

13. Write a short algorithm that describes how to solve the system $Y' = AY$ if A is not diagonalizable.

14. Set up the initial value problem that describes the following scenario:

Two tanks are connected by a series of pipes as shown below. Tank 1 initially contains 150 grams of salt dissolved in 20 liters of water, while tank 2 contains 50 grams of salt dissolved in 10 liters of water.

Beginning at time t = 0, pure water is pumped into tank 1 at a volumetric flow rate of 3 liters per minute. This causes brine to flow between the tanks, and to flow out of both tanks at the volumetric flow rates shown in the Fig. 2. Determine the salt concentration in each tank (C_1 and C_2) at any time t > 0.