5.4

#6 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -2 \\ -3 & \lambda-4 \end{vmatrix} = (\lambda-1)(\lambda-4) - 6$$

$$= \lambda^2 - 5\lambda - 2$$

$$\lambda = \frac{5 \pm \sqrt{25 - 4\cdot 1(-2)}}{2} = \frac{5 \pm \sqrt{33}}{2}$$

$\lambda = \frac{5+\sqrt{33}}{2}$:
$$\left[\begin{array}{cc|c} \frac{5+\sqrt{33}}{2} - 1 & -2 & 0 \\ -3 & \frac{5+\sqrt{33}}{2} - 4 & 0 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} \frac{\sqrt{33}+3}{2} & -2 & 0 \\ -3 & \frac{\sqrt{33}-3}{2} & 0 \end{array}\right] \quad R_1 \Big/ \left(\frac{\sqrt{33}+3}{2}\right)$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -\frac{(\sqrt{33}-3)}{6} & 0 \\ -3 & \frac{\sqrt{33}-3}{2} & 0 \end{array}\right] \quad 3R_1 + R_2$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -\left(\frac{\sqrt{33}-3}{6}\right) & 0 \\ 0 & 0 & 0 \end{array}\right]$$

y is free and $x = \frac{\sqrt{33}-3}{6} y$

$$\begin{bmatrix} x \\ y \end{bmatrix} \in E_{\frac{5+\sqrt{33}}{2}} \iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{33}-3}{6} y \\ y \end{bmatrix} = \frac{y}{6} \begin{bmatrix} \sqrt{33}-3 \\ 6 \end{bmatrix}$$

$\therefore \begin{bmatrix} \sqrt{33}-3 \\ 6 \end{bmatrix}$ is a basis for $E_{\frac{5+\sqrt{33}}{2}}$

$\lambda = \frac{5-\sqrt{33}}{2}$: $\left[\begin{array}{cc|c} \frac{5-\sqrt{33}}{2}-1 & -2 & 0 \\ -3 & \frac{5-\sqrt{33}}{2}-4 & 0 \end{array}\right]$

$$= \left[\begin{array}{cc|c} \frac{3-\sqrt{33}}{2} & -2 & 0 \\ -3 & \frac{-3-\sqrt{33}}{2} & 0 \end{array}\right] R_1 \Big/ \left(\frac{3-\sqrt{33}}{2}\right)$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & \frac{3+\sqrt{33}}{6} & 0 \\ -3 & -\left(\frac{3+\sqrt{33}}{2}\right) & 0 \end{array}\right] 3R_1 + R_2$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & \frac{3+\sqrt{33}}{6} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

y free , $x = -\left(\frac{3+\sqrt{33}}{6}\right)y$

$$\begin{bmatrix} x \\ y \end{bmatrix} \in E_{\frac{5-\sqrt{33}}{2}} \iff \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\left(\frac{3+\sqrt{33}}{6}\right)y \\ y \end{bmatrix}$$

$$= -\frac{y}{6}\begin{bmatrix} 3+\sqrt{33} \\ -6 \end{bmatrix}$$

$\therefore \begin{bmatrix} \sqrt{33}+3 \\ -6 \end{bmatrix}$ is a basis for $E_{\frac{5-\sqrt{33}}{2}}$.

#18 $\quad A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-3 & -1 & 1 \\ 0 & \lambda-0 & 2 \\ 0 & -1 & \lambda-2 \end{vmatrix}$$

$$= (\lambda-3)\cdot\left[\lambda(\lambda-2)+2\right]$$

$$= (\lambda-3)(\lambda^2-2\lambda+2)$$ quadratic formula.

$$= (\lambda-3)(\lambda-(1+i))(\lambda - (1-i))$$

$\lambda = 3$: $\quad \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} f & d & d \end{array} \end{array}$

x is free and $y=0$ and $z=0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_3 \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is a basis for E_3

$$\lambda = 1+i: \quad \left[\begin{array}{ccc|c} -2+i & -1 & 1 & 0 \\ 0 & 1+i & 2 & 0 \\ 0 & -1 & -1+i & 0 \end{array}\right] \begin{array}{l} R_1 - R_3 \\ (1+i)R_3 + R_2 \\ \ \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} -2+i & 0 & 2-i & 0 \\ 0 & 0 & 2+(-1)(1-i)(1+i) & 0 \\ 0 & -1 & -1+i & 0 \end{array}\right] \begin{array}{l} R_1/(-2+i) \\ \ \\ R_3/-1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1-i & 0 \end{array}\right]$$

$$\frac{2-i}{-2+i} \cdot \frac{-2-i}{-2-i} = \frac{-1(2-i)(2+i)}{(-2+i)(-2-i)} = -1 \cdot \frac{(4+1)}{4+1} = -1$$

z is free, $x = z$ and $y = -(1-i)z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_{1+i} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ -(1-i)z \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ -1+i \\ 1 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ -1+i \\ 1 \end{bmatrix}$ is a basis for E_{1+i}

No need to also calculate a basis for E_{1+i}

simply conjugate.

$$\overline{\begin{bmatrix} 1 \\ -1+i \\ 1 \end{bmatrix}} = \begin{bmatrix} 1 \\ -1-i \\ 1 \end{bmatrix} \text{ is a basis for } E_{1-i}$$