5.6

#6. $D^2 : V \to V$ where V is the vector space of solutions to the differential eq'n $y'' + y = 0$.

(i) Find a basis for V.:

the characteristic poly of $y'' + y = 0$ is

$$p(\lambda) = \lambda^2 + 1$$

roots: $\pm i$.

pick $r = +i$. Then e^{ix} is a complex solution.

$$e^{ix} = \cos x + i \sin x$$

$\Rightarrow$ $y_1 = \cos x$ & $y_2 = \sin x$ are real solutions.

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

$\neq 0$ everywhere.

$\Rightarrow$ y_1 & y_2 are L.I in V

$\Rightarrow$ since $\dim V = 2$, then y_1 and y_2 are a basis.

Set $\alpha = \{\cos x, \sin x\}$.

(ii) Find $[D^2]_\alpha^\alpha =: A$.

$$A = [D^2]_\alpha^\alpha = \Big[[D^2 \cos x]_\alpha \quad [D^2 \sin x]_\alpha \Big]$$

$$= \Big[[-\cos x]_\alpha \quad [-\sin x]_\alpha \Big]$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(iii) Find eigenvalues of A & bases for eigenspaces:

A is diagonal! we can read the eigenvalues off the matrix

eigenvalues $\lambda = -1$ (mult 2)

$\lambda = -1$: $\begin{bmatrix} \lambda+1 & 0 \\ 0 & \lambda+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

both x & y are free

$$\begin{bmatrix} x \\ y \end{bmatrix} \in E_{-1} \iff \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a basis for E_{-1}.

(iv) Send this data back to V using the map

$$\varphi : \mathbb{R}^2 \longrightarrow V$$

$$\varphi\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_1 \cos x + c_2 \sin x$$

(a) -1 is an eigenvalue of D^2.

(b) $\varphi\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos x$ & $\varphi\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sin x$ are a basis for V_{-1}.

(b) D^2 is diagonalizable since $[D^2]_\alpha^\alpha$ is,

$\dim E_{-1} = 2.$