6.2

#4. Find the general solution of

$$Y' = \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} Y$$

From 5.4 (see solutions) we know

$$D = P^{-1} A P$$

where

$$D = \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 0 \\ 0 & \frac{5-\sqrt{33}}{2} \end{bmatrix}$$ and

$$P = \begin{bmatrix} \sqrt{33}-3 & \sqrt{33}+3 \\ 6 & -6 \end{bmatrix}$$

For convenience let $s = \frac{5+\sqrt{33}}{2}$ and $t = \frac{5-\sqrt{33}}{2}$.

Then the general solution to

$$Y' = DY \text{ ,}$$

is

$$Z = \begin{bmatrix} c_1 e^{sx} \\ c_2 e^{tx} \end{bmatrix}$$

then the general solution to

$$Y' = AY$$

is

only $\sqrt{33}$

$$Y_H = PZ = \begin{bmatrix} \sqrt{33}-3 & \sqrt{33}+3 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} c_1 e^{sx} \\ c_2 e^{tx} \end{bmatrix}$$

$$= \begin{bmatrix} c_1\left(\sqrt{33}-3\right)e^{sx} + c_2\left(\sqrt{33}+3\right)e^{tx} \\ 6c_1 e^{sx} - 6c_2 e^{tx} \end{bmatrix}$$

#12. Find the general solution to

$$Y' = \underbrace{\begin{bmatrix} 3 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}}_{A} Y$$

From 5.4 (see solutions) we know

$$D = P^{-1} A P$$

where

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1+i & -1-i \\ 0 & 1 & 1 \end{bmatrix}$$

The general solution to

$$Y' = DY$$

is

$$\begin{bmatrix} c_1 e^{3x} \\ c_2 e^{(1+i)x} \\ c_3 e^{(1-i)x} \end{bmatrix} = c_1 \begin{bmatrix} e^{3x} \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{(1+i)x} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ e^{(1-i)x} \end{bmatrix}$$

From $\begin{bmatrix} e^{3x} \\ 0 \\ 0 \end{bmatrix}$ we obtain the solution (of $Y' = AY$)

$$Y_1 = P \begin{bmatrix} e^{3x} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1+i & -1-i \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3x} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{3x} \\ 0 \\ 0 \end{bmatrix}$$

Next, obtain two "real-valued" solutions (of $Y' = AY$) from the "complex-valued" solution (of $Y' = DY$)

$$\begin{bmatrix} 0 \\ e^{(1+i)x} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ e^x \cos x + i \cdot e^x \sin x \\ 0 \end{bmatrix}$$

$$\Rightarrow P \begin{bmatrix} 0 \\ e^x \cos x + i e^x \sin x \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1+i & -1-i \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ e^x \cos x + i e^x \sin x \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^x \cos x + i e^x \sin x \\ (-1+i)(e^x \cos x + i e^x \sin x) \\ e^x \cos x + i \cdot e^x \sin x \end{bmatrix}$$

$$(-1+i)(e^x\cos x + i e^x \sin x)$$

$$= e^x \cdot \left[-\cos x - i\sin x + i\cos x + i^2 \sin x \right]$$

$$= e^x \left[(-\cos x - \sin x) + i(\cos x - \sin x) \right]$$

$$= e^x(-\cos x - \sin x) + i \cdot e^x(\cos x - \sin x)$$

$$\Rightarrow P \begin{bmatrix} 0 \\ e^x\cos x + i e^x \sin x \\ 0 \end{bmatrix} = \begin{bmatrix} e^x \cos x + i \cdot e^x \sin x \\ e^x(-\cos x - \sin x) + i \cdot e^x(\cos x - \sin x) \\ e^x\cos x + i \cdot e^x \sin x \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^x\cos x \\ e^x(-\cos x - \sin x) \\ e^x \cos x \end{bmatrix}}_{Y_2} + i \underbrace{\begin{bmatrix} e^x \sin x \\ e^x(\cos x - \sin x) \\ e^x \sin x \end{bmatrix}}_{Y_3}$$

Claim: Y_1, Y_2, Y_3 are a fundamental set of solutions of $Y' = AY$.

It suffices to demonstrate the vectors are L.I.

Compute the Wronskian

$$W\big(y_1(x), y_2(x), y_3(x)\big)$$

$$= \begin{vmatrix} e^{3x} & e^x \cos x & e^x \sin x \\ 0 & e^x(-\cos x - \sin x) & e^x(\cos x - \sin x) \\ 0 & e^x \cos x & e^x \sin x \end{vmatrix}$$

$$= e^{3x} \cdot \begin{vmatrix} e^x(-\cos x - \sin x) & e^x(\cos x - \sin x) \\ e^x \cos x & e^x \sin x \end{vmatrix}$$

$$= e^{3x} \cdot (e^x)^2 \cdot \begin{vmatrix} -\cos x - \sin x & \cos x - \sin x \\ \cos x & \sin x \end{vmatrix}$$

$R_1 \leftarrow R_1 + R_2$ (theorem 1.32.3)

$$= e^{3x} (e^x)^2 \begin{vmatrix} -\sin x & \cos x \\ \cos x & \sin x \end{vmatrix}$$

$$= e^{5x} \cdot \left(-\sin^2 x - \cos^2 x\right)$$

$$= -e^{5x}$$

$$\neq 0.$$

Thus, the general solution to $Y' = AY$ is

$$Y_H = c_1 Y_1 + c_2 Y_2 + c_3 Y_3$$

$$= \begin{bmatrix} c_1 e^{3x} + c_2 e^x \cos x + c_3 e^x \sin x \\ c_2 e^x(-\cos x - \sin x) + c_3 e^x(\cos x - \sin x) \\ c_2 e^x \cos x + c_3 e^x \sin x \end{bmatrix}$$