#16.3.16

$$\vec{F} = \langle y^2z + 2xz^2,\ 2xyz,\ xy^2 + 2x^2z \rangle$$

$$C:\ \vec{r}(t) = \langle \sqrt{t},\ t+1,\ t^2 \rangle, \quad 0 \le t \le 1$$

① $f_x = y^2z + 2xz^2$

② $f_y = 2xyz$

③ $f_z = xy^2 + 2x^2z$

$$f(x,y,z) = \int f_x\,dx \overset{①}{=} \int y^2z + 2xz^2\,dx$$

$$\Rightarrow f(x,y,z) = xy^2z + x^2z^2 + g(y,z)$$

$$\Rightarrow f_y = \frac{\partial}{\partial y}\left(xy^2z + x^2z^2 + g(y,z)\right) \overset{②}{=} 2xyz$$

$$\Rightarrow 2xyz + \frac{\partial g}{\partial y} = 2xyz$$

$$\Rightarrow g_y(y,z) = 0$$

$$\Rightarrow g(y,z) = \int g_y\,dy = h(z)$$

$$\Rightarrow f(x,y,z) = xy^2z + x^2z^2 + h(z)$$

3

$$\Rightarrow f_z = \frac{\partial}{\partial z}\left(xy^2z + x^2z^2 + h(z)\right) = xy^2 + 2x^2z$$

$$xy^2 + 2x^2z + h'(z) = xy^2 + 2x^2z$$

$$\Rightarrow h'(z) = 0$$

$$\Rightarrow h(z) = C$$

$$\therefore f(x,y,z) = xy^2z + x^2z^2 + C$$

(b) $$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

$$= f(\vec{r}(1)) - f(\vec{r}(0))$$

$$= f(1,2,1) - f(0,1,0)$$

$$= (4 + 1 + C) - (0 + 0 + C)$$

$$= 5$$

#16.3.20 Find ~~Show~~ $\displaystyle\int_C \overbrace{\sin y}^{P}\,dx + \overbrace{(x\cos y - \sin y)}^{Q}\,dy$

where C is any path from $(2,0)$ to $(1,\pi)$

$\vec{F} = \langle P, Q \rangle \Rightarrow \operatorname{curl}\vec{F} = Q_x - P_y$

$= \cos y - \cos y$

$= 0$

$\therefore\ \vec{F}$ is conservative

$\Rightarrow \displaystyle\int_C \vec{F}\cdot d\vec{r}$ is path independent.

Find f s.t. $\nabla f = \vec{F}$.

① $f_x = \sin y$

② $f_y = x\cos y - \sin y$

$\longrightarrow$

$$f(x,y) = \int f_x \, dx \overset{①}{=} \int \sin y \, dx = x \sin y + g(y)$$

$$\Rightarrow f_y = \frac{\partial}{\partial y}\left(x \sin y + g(y)\right) \overset{②}{=} x\cos y - \sin y$$

$$x \cos y + g'(y) = x \cos y - \sin y$$

$$\Rightarrow \quad g'(y) = -\sin y$$

$$\Rightarrow g(y) = \int -\sin y \, dy = \cos y + C$$

$$\therefore f(x,y) = x \sin y + \cos y + C$$

$$\therefore \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(1,\pi) - f(2,0)$$

$$= \left(0 + (-1) + C\right) - \left(0 + 1 + C\right)$$

$$= -2$$

$$C_1 : \quad \vec{r}(t) = \langle \cos t, \sin t \rangle, \qquad 0 \le t \le \pi$$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

$$\vec{F}(\vec{r}(t)) = \frac{-\sin t\, \hat{i} + \cos t\, \hat{j}}{\cos^2 t + \sin^2 t} = \langle -\sin t, \cos t \rangle$$

$$\vec{F}(\vec{r}(t)) = \sin^2 t + \cos^2 t = 1.$$
$$\cdot\ \vec{r}\,'(t).$$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^{\pi} 1 \, dt = \pi.$$

$$C_2 : \quad \vec{r}(t) = \langle \cos t, \sin t \rangle, \qquad \pi \le t \le 2\pi$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{\pi}^{2\pi} 1 \, dt = \pi$$

16.3.35 $\vec{F} = \dfrac{-y\,\hat{i} + x\,\hat{j}}{x^2+y^2}$

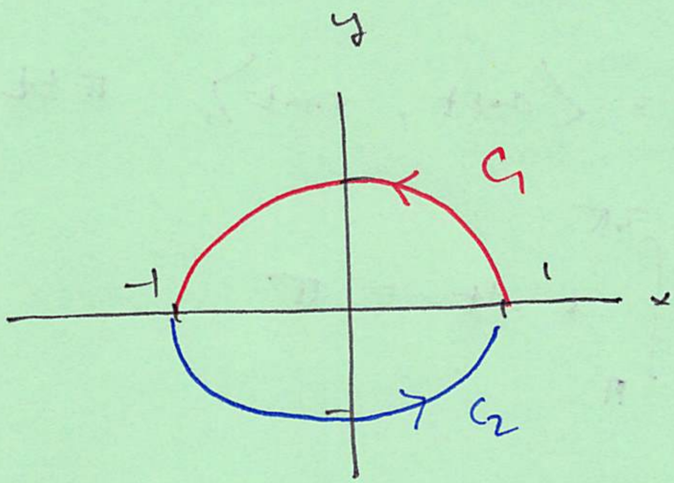
(a) $\text{curl}\,\vec{F} = \dfrac{\partial}{\partial x}\left[\dfrac{x}{x^2+y^2}\right] - \dfrac{\partial}{\partial y}\left[\dfrac{-y}{x^2+y^2}\right]$

$$= \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2 + x^2 - y^2}{(x^2+y^2)^2}$$

$$= 0.$$

(b)



Hence, $-C_2$ and C_1 are two paths from $(1,0)$ to $(-1,0)$; on the one hand

$$\int_{C_1} \vec{F} \cdot dr = \pi,$$

and on the other

$$\int_{-C_2} \vec{F} \cdot d\vec{r} = -\int_{C_2} \vec{F} \cdot d\vec{r} = -\pi$$

$\therefore \int_C \vec{F} \cdot d\vec{r}$ is not independent of path.

This doesn't contradict theorem 6 since $\vec{F}$ is not C^1 at $(0,0)$ (it's not even defined there). The theorem doesn't apply.

$\# 16.5.16 \quad \vec{F} = \langle 1, \sin z, y\cos z \rangle$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ D_x & D_y & D_z \\ 1 & \sin z & y\cos z \end{vmatrix}$$

$$= \hat{i}(\cos z - \cos z) - \hat{j}(0 - 0) + \hat{k}(0 - 0)$$

$$= \vec{0}$$

$\therefore \vec{F}$ is conservative.

Find f s.t. $\nabla f = \vec{F}$

① $f_x = 1$

② $f_y = \sin z$

③ $f_z = y\cos z$.

1

$$f(x,y,z) = \int f_x \, dx = \int 1 \, dx = x + g(y,z)$$

2

$$\Rightarrow f_y = \frac{\partial}{\partial y}\left(x + g(y,z)\right) = \sin z$$

$$g_y(y,z) = \sin z$$

$$\Rightarrow g(y,z) = \int g_z = \int \sin z = -\cos z + h(z)$$

$$\therefore f(x,y,z) = x - \cos z + h(z)$$

3

$$\Rightarrow f_z = \frac{\partial}{\partial z}\left(x - \cos z + h(z)\right) = y \cos z$$

$$\sin z + h'(y) = y \cos z$$

$$h'(z) = y \cos z - \sin z$$

$$\Rightarrow h(z) = \int y \cos z - \sin z \, dz$$

$$= y \sin z + \cos z + C.$$

$$\therefore f(x,y,z) = x + y \sin z + C.$$