1.2 Matrices

If we examine the method of elimination described in Section 1.1, we can make the following observation: Only the numbers in front of the unknowns $x_1, x_2, \ldots, x_n$ and the numbers $b_1, b_2, \ldots, b_m$ on the right side are being changed as we perform the steps in the method of elimination. Thus we might think of looking for a way of writing a linear system without having to carry along the unknowns. Matrices enable us to do this—that is, to write linear systems in a compact form that makes it easier to automate the elimination method by using computer software in order to obtain a fast and efficient procedure for finding solutions. The use of matrices, however, is not merely that of a convenient notation. We now develop operations on matrices and will work with matrices according to the rules they obey; this will enable us to solve systems of linear equations and to handle other computational problems in a fast and efficient manner. Of course, as any good definition should do, the notion of a matrix not only provides a new way of looking at old problems, but also gives rise to a great many new questions, some of which we study in this book.

DEFINITION 1.1

An $m \times n$ **matrix** A is a rectangular array of mn real or complex numbers arranged in m horizontal **rows** and n vertical **columns**:

$$A = \begin{array}{cl} \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & a_{ij} & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & \cdots & a_{mn} \end{bmatrix} . & \leftarrow i\text{th row} \\ \uparrow\, j\text{th column} & \end{array} \qquad (1)$$

The ***i*th row** of A is

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \qquad (1 \leq i \leq m);$$

the ***j*th column** of A is

$$\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \qquad (1 \leq j \leq n).$$

We shall say that A is ***m* by *n*** (written as $\boldsymbol{m \times n}$). If $m = n$, we say that A is a **square matrix of order *n***, and that the numbers $a_{11}, a_{22}, \ldots, a_{nn}$ form the **main diagonal** of A. We refer to the number a_{ij}, which is in the ith row and jth column of A, as the ***i*, *j*th element** of A, or the **(*i*, *j*) entry** of A, and we often write (1) as

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}.$$

EXAMPLE 1

Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1+i & 4i \\ 2-3i & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 3 \end{bmatrix}, \quad F = \begin{bmatrix} -1 & 0 & 2 \end{bmatrix}.$$

Then A is a 2×3 matrix with $a_{12} = 2$, $a_{13} = 3$, $a_{22} = 0$, and $a_{23} = 1$; B is a 2×2 matrix with $b_{11} = 1+i$, $b_{12} = 4i$, $b_{21} = 2-3i$, and $b_{22} = -3$; C is a 3×1 matrix with $c_{11} = 1$, $c_{21} = -1$, and $c_{31} = 2$; D is a 3×3 matrix; E is a 1×1 matrix; and F is a 1×3 matrix. In D, the elements $d_{11} = 1$, $d_{22} = 0$, and $d_{33} = 2$ form the main diagonal. ■

For convenience, we focus much of our attention in the illustrative examples and exercises in Chapters 1–6 on matrices and expressions containing only real numbers. Complex numbers make a brief appearance in Chapter 7. An introduction to complex numbers, their properties, and examples and exercises showing how complex numbers are used in linear algebra may be found in Appendix B.

An $n \times 1$ matrix is also called an ***n*-vector** and is denoted by lowercase boldface letters. When n is understood, we refer to n-vectors merely as **vectors**. Vectors are discussed at length in Section 4.1.

EXAMPLE 2

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \text{ is a 4-vector and } \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \text{ is a 3-vector.}$$ ■

The n-vector all of whose entries are zero is denoted by $\mathbf{0}$.

Observe that if A is an $n \times n$ matrix, then the rows of A are $1 \times n$ matrices and the columns of A are $n \times 1$ matrices. The set of all n-vectors with real entries is denoted by R^n. Similarly, the set of all n-vectors with complex entries is denoted by C^n. As we have already pointed out, in the first six chapters of this book we work almost entirely with vectors in R^n.

EXAMPLE 3

(Tabular Display of Data) The following matrix gives the airline distances between the indicated cities (in statute miles):

$$\begin{array}{l} \\ \text{London} \\ \text{Madrid} \\ \text{New York} \\ \text{Tokyo} \end{array} \begin{array}{c} \begin{array}{cccc} \text{London} & \text{Madrid} & \text{New York} & \text{Tokyo} \end{array} \\ \begin{bmatrix} 0 & 785 & 3469 & 5959 \\ 785 & 0 & 3593 & 6706 \\ 3469 & 3593 & 0 & 6757 \\ 5959 & 6706 & 6757 & 0 \end{bmatrix} \end{array}$$ ■

EXAMPLE 4

(Production) Suppose that a manufacturer has four plants, each of which makes three products. If we let a_{ij} denote the number of units of product i made by plant

j in one week, then the 3×4 matrix

$$\begin{array}{r} \\ \text{Product 1} \\ \text{Product 2} \\ \text{Product 3} \end{array} \begin{array}{c} \begin{array}{cccc} \text{Plant 1} & \text{Plant 2} & \text{Plant 3} & \text{Plant 4} \end{array} \\ \begin{bmatrix} 560 & 360 & 380 & 0 \\ 340 & 450 & 420 & 80 \\ 280 & 270 & 210 & 380 \end{bmatrix} \end{array}$$

gives the manufacturer's production for the week. For example, plant 2 makes 270 units of product 3 in one week. ■

EXAMPLE 5

The windchill table that follows is a matrix.

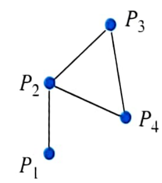
			°F			
mph	15	10	5	0	−5	−10
5	12	7	0	−5	−10	−15
10	−3	−9	−15	−22	−27	−34
15	−11	−18	−25	−31	−38	−45
20	−17	−24	−31	−39	−46	−53

A combination of air temperature and wind speed makes a body feel colder than the actual temperature. For example, when the temperature is 10°F and the wind is 15 miles per hour, this causes a body heat loss equal to that when the temperature is −18°F with no wind. ■

EXAMPLE 6

By a **graph** we mean a set of points called **nodes** or **vertices**, some of which are connected by lines called **edges**. The nodes are usually labeled as $P_1, P_2, \ldots, P_k$, and for now we allow an edge to be traveled in either direction. One mathematical representation of a graph is constructed from a table. For example, the following table represents the graph shown:

	P_1	P_2	P_3	P_4
P_1	0	1	0	0
P_2	1	0	1	1
P_3	0	1	0	1
P_4	0	1	1	0



The (i, j) entry $= 1$ if there is an edge connecting vertex P_i to vertex P_j; otherwise, the (i, j) entry $= 0$. The **incidence matrix** A is the $k \times k$ matrix obtained by omitting the row and column labels from the preceding table. The incidence matrix for the corresponding graph is

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

■

Internet search engines use matrices to keep track of the locations of information, the type of information at a location, keywords that appear in the information, and even the way websites link to one another. A large measure of the effectiveness of the search engine Google© is the manner in which matrices are used to determine which sites are referenced by other sites. That is, instead of directly keeping track of the information content of an actual web page or of an individual search topic, Google's matrix structure focuses on finding web pages that match the search topic, and then presents a list of such pages in the order of their "importance."

Suppose that there are n accessible web pages during a certain month. A simple way to view a matrix that is part of Google's scheme is to imagine an $n \times n$ matrix A, called the "connectivity matrix," that initially contains all zeros. To build the connections, proceed as follows. When you detect that website j links to website i, set entry a_{ij} equal to one. Since n is quite large, in the billions, most entries of the connectivity matrix A are zero. (Such a matrix is called sparse.) If row i of A contains many ones, then there are many sites linking to site i. Sites that are linked to by many other sites are considered more "important" (or to have a higher rank) by the software driving the Google search engine. Such sites would appear near the top of a list returned by a Google search on topics related to the information on site i. Since Google updates its connectivity matrix about every month, n increases over time and new links and sites are adjoined to the connectivity matrix.

In Chapter 8 we elaborate a bit on the fundamental technique used for ranking sites and give several examples related to the matrix concepts involved. Further information can be found in the following sources:

1. Berry, Michael W., and Murray Browne. *Understanding Search Engines—Mathematical Modeling and Text Retrieval*, 2d ed. Philadelphia: Siam, 2005.
2. www.google.com/technology/index.html
3. Moler, Cleve. "The World's Largest Matrix Computation: Google's PageRank Is an Eigenvector of a Matrix of Order 2.7 Billion," MATLAB *News and Notes*, October 2002, pp. 12–13.

Whenever a new object is introduced in mathematics, we must determine when two such objects are equal. For example, in the set of all rational numbers, the numbers $\frac{2}{3}$ and $\frac{4}{6}$ are called equal, although they have different representations. What we have in mind is the definition that a/b equals c/d when $ad = bc$. Accordingly, we now have the following definition:

DEFINITION 1.2 Two $m \times n$ matrices $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ are **equal** if they agree entry by entry, that is, if $a_{ij} = b_{ij}$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

EXAMPLE 7 The matrices

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & z \end{bmatrix}$$

are equal if and only if $w = -1$, $x = -3$, $y = 0$, and $z = 5$. ■

■ Matrix Operations

We next define a number of operations that will produce new matrices out of given matrices. When we are dealing with linear systems, for example, this will enable us to manipulate the matrices that arise and to avoid writing down systems over and over again. These operations and manipulations are also useful in other applications of matrices.

Matrix Addition

DEFINITION 1.3

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ are both $m \times n$ matrices, then the **sum** $A+B$ is an $m \times n$ matrix $C = \begin{bmatrix} c_{ij} \end{bmatrix}$ defined by $c_{ij} = a_{ij} + b_{ij}$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. Thus, to obtain the sum of A and B, we merely add corresponding entries.

EXAMPLE 8

Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 3 & -4 \end{bmatrix}.$$

Then

$$A + B = \begin{bmatrix} 1+0 & -2+2 & 3+1 \\ 2+1 & -1+3 & 4+(-4) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & 0 \end{bmatrix}.$$

■

EXAMPLE 9

(Production) A manufacturer of a certain product makes three models, A, B, and C. Each model is partially made in factory F_1 in Taiwan and then finished in factory F_2 in the United States. The total cost of each product consists of the manufacturing cost and the shipping cost. Then the costs at each factory (in dollars) can be described by the 3×2 matrices F_1 and F_2:

$$F_1 = \begin{array}{c} \begin{array}{cc} \text{Manufacturing cost} & \text{Shipping cost} \end{array} \\ \begin{bmatrix} 32 & 40 \\ 50 & 80 \\ 70 & 20 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array}$$

$$F_2 = \begin{array}{c} \begin{array}{cc} \text{Manufacturing cost} & \text{Shipping cost} \end{array} \\ \begin{bmatrix} 40 & 60 \\ 50 & 50 \\ 130 & 20 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Model A} \\ \text{Model B} \\ \text{Model C} \end{array}.$$

The matrix $F_1 + F_2$ gives the total manufacturing and shipping costs for each product. Thus the total manufacturing and shipping costs of a model C product are \$200 and \$40, respectively. ■

If $\mathbf{x}$ is an n-vector, then it is easy to show that $\mathbf{x} + \mathbf{0} = \mathbf{x}$, where $\mathbf{0}$ is the n-vector all of whose entries are zero. (See Exercise 16.)

It should be noted that the sum of the matrices A and B is defined only when A and B have the same number of rows and the same number of columns, that is, only when A and B are of the same size.

We now make the convention that when $A + B$ is written, both A and B are of the same size.

The basic properties of matrix addition are considered in the next section and are similar to those satisfied by the real numbers.

Scalar Multiplication

DEFINITION 1.4

If $A = \left[a_{ij}\right]$ is an $m \times n$ matrix and r is a real number, then the **scalar multiple** of A by r, rA, is the $m \times n$ matrix $C = \left[c_{ij}\right]$, where $c_{ij} = ra_{ij}$, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$; that is, the matrix C is obtained by multiplying each entry of A by r.

EXAMPLE 10

We have

$$-2\begin{bmatrix} 4 & -2 & -3 \\ 7 & -3 & 2 \end{bmatrix} = \begin{bmatrix} (-2)(4) & (-2)(-2) & (-2)(-3) \\ (-2)(7) & (-2)(-3) & (-2)(2) \end{bmatrix}$$
$$= \begin{bmatrix} -8 & 4 & 6 \\ -14 & 6 & -4 \end{bmatrix}.$$ ■

Thus far, addition of matrices has been defined for only two matrices. Our work with matrices will call for adding more than two matrices. Theorem 1.1 in Section 1.4 shows that addition of matrices satisfies the associative property: $A + (B + C) = (A + B) + C$.

If A and B are $m \times n$ matrices, we write $A + (-1)B$ as $A - B$ and call this the **difference between A and B**.

EXAMPLE 11

Let

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 4 & 2 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 5 & -2 \end{bmatrix}.$$

Then

$$A - B = \begin{bmatrix} 2-2 & 3+1 & -5-3 \\ 4-3 & 2-5 & 1+2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & -8 \\ 1 & -3 & 3 \end{bmatrix}.$$ ■

Application

Vectors in R^n can be used to handle large amounts of data. Indeed, a number of computer software products, notably, MATLAB®, make extensive use of vectors. The following example illustrates these ideas:

EXAMPLE 12

(Inventory Control) Suppose that a store handles 100 different items. The inventory on hand at the beginning of the week can be described by the inventory vector $\mathbf{u}$ in R^{100}. The number of items sold at the end of the week can be described by the 100-vector $\mathbf{v}$, and the vector

$$\mathbf{u} - \mathbf{v}$$

represents the inventory at the end of the week. If the store receives a new shipment of goods, represented by the 100-vector $\mathbf{w}$, then its new inventory would be

$$\mathbf{u} - \mathbf{v} + \mathbf{w}.$$ ■

We shall sometimes use the **summation notation**, and we now review this useful and compact notation.

By $\sum_{i=1}^{n} a_i$ we mean $a_1 + a_2 + \cdots + a_n$. The letter i is called the **index of summation**; it is a dummy variable that can be replaced by another letter. Hence we can write

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k.$$

Thus

$$\sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4.$$

The summation notation satisfies the following properties:

1. $\sum_{i=1}^{n}(r_i + s_i)a_i = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} s_i a_i$
2. $\sum_{i=1}^{n} c(r_i a_i) = c \sum_{i=1}^{n} r_i a_i$
3. $\sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij}\right) = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\right)$

Property 3 can be interpreted as follows: The left side is obtained by adding all the entries in each column and then adding all the resulting numbers. The right side is obtained by adding all the entries in each row and then adding all the resulting numbers.

If $A_1, A_2, \ldots, A_k$ are $m \times n$ matrices and $c_1, c_2, \ldots, c_k$ are real numbers, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \cdots + c_k A_k \tag{2}$$

is called a **linear combination** of $A_1, A_2, \ldots, A_k$, and $c_1, c_2, \ldots, c_k$ are called **coefficients.**

The linear combination in Equation (2) can also be expressed in summation notation as

$$\sum_{i=1}^{k} c_i A_i = c_1 A_1 + c_2 A_2 + \cdots + c_k A_k.$$

EXAMPLE 13

The following are linear combinations of matrices:

$$3\begin{bmatrix} 0 & -3 & 5 \\ 2 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 5 & 2 & 3 \\ 6 & 2 & 3 \\ -1 & -2 & 3 \end{bmatrix},$$

$$2\begin{bmatrix} 3 & -2 \end{bmatrix} - 3\begin{bmatrix} 5 & 0 \end{bmatrix} + 4\begin{bmatrix} -2 & 5 \end{bmatrix},$$

$$-0.5\begin{bmatrix} 1 \\ -4 \\ -6 \end{bmatrix} + 0.4\begin{bmatrix} 0.1 \\ -4 \\ 0.2 \end{bmatrix}.$$

Using scalar multiplication and matrix addition, we can compute each of these linear combinations. Verify that the results of such computations are, respectively,

$$\begin{bmatrix} -\frac{5}{2} & -10 & \frac{27}{2} \\ 3 & 8 & \frac{21}{2} \\ \frac{7}{2} & -5 & -\frac{21}{2} \end{bmatrix}, \qquad \begin{bmatrix} -17 & 16 \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} -0.46 \\ 0.4 \\ 3.08 \end{bmatrix}.$$

EXAMPLE 14 Let

$$\mathbf{p} = \begin{bmatrix} 18.95 \\ 14.75 \\ 8.60 \end{bmatrix}$$

be a 3-vector that represents the current prices of three items at a store. Suppose that the store announces a sale so that the price of each item is reduced by 20%.

(a) Determine a 3-vector that gives the price changes for the three items.
(b) Determine a 3-vector that gives the new prices of the items.

Solution

(a) Since each item is reduced by 20%, the 3-vector

$$-0.20\mathbf{p} = \begin{bmatrix} (-0.20)18.95 \\ (-0.20)14.75 \\ (-0.20)8.60 \end{bmatrix} = \begin{bmatrix} -3.79 \\ -2.95 \\ -1.72 \end{bmatrix} = -\begin{bmatrix} 3.79 \\ 2.95 \\ 1.72 \end{bmatrix}$$

gives the price changes for the three items.

(b) The new prices of the items are given by the expression

$$\mathbf{p} - 0.20\mathbf{p} = \begin{bmatrix} 18.95 \\ 14.75 \\ 8.60 \end{bmatrix} - \begin{bmatrix} 3.79 \\ 2.95 \\ 1.72 \end{bmatrix} = \begin{bmatrix} 15.16 \\ 11.80 \\ 6.88 \end{bmatrix}.$$

Observe that this expression can also be written as

$$\mathbf{p} - 0.20\mathbf{p} = 0.80\mathbf{p}.$$

■

The next operation on matrices is useful in a number of situations.

DEFINITION 1.5 If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is an $m \times n$ matrix, then the **transpose** of A, $A^T = \begin{bmatrix} a_{ij}^T \end{bmatrix}$, is the $n \times m$ matrix defined by $a_{ij}^T = a_{ji}$. Thus the transpose of A is obtained from A by interchanging the rows and columns of A.

EXAMPLE 15 Let

$$A = \begin{bmatrix} 4 & -2 & 3 \\ 0 & 5 & -2 \end{bmatrix}, \qquad B = \begin{bmatrix} 6 & 2 & -4 \\ 3 & -1 & 2 \\ 0 & 4 & 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 5 & 4 \\ -3 & 2 \\ 2 & -3 \end{bmatrix},$$

$$D = \begin{bmatrix} 3 & -5 & 1 \end{bmatrix}, \qquad E = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} 4 & 0 \\ -2 & 5 \\ 3 & -2 \end{bmatrix}, \qquad B^T = \begin{bmatrix} 6 & 3 & 0 \\ 2 & -1 & 4 \\ -4 & 2 & 3 \end{bmatrix},$$

$$C^T = \begin{bmatrix} 5 & -3 & 2 \\ 4 & 2 & -3 \end{bmatrix}, \quad D^T = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}, \quad \text{and} \quad E^T = \begin{bmatrix} 2 & -1 & 3 \end{bmatrix}.$$

■

Key Terms

Matrix
Rows
Columns
Size of a matrix
Square matrix
Main diagonal
Element or entry of a matrix
Equal matrices
n-vector (or vector)
R^n, C^n
0, zero vector
Google
Matrix addition
Scalar multiple
Difference of matrices
Summation notation
Index of summation
Linear combination
Coefficients
Transpose

1.2 Exercises

1. Let

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 6 & -5 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ -3 \\ 5 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 7 & 3 & 2 \\ -4 & 3 & 5 \\ 6 & 1 & -1 \end{bmatrix}.$$

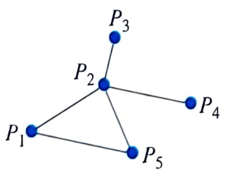
(a) What is a_{12}, a_{22}, a_{23}?

(b) What is b_{11}, b_{31}?

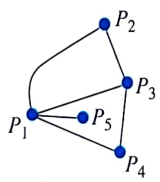
(c) What is c_{13}, c_{31}, c_{33}?

2. Determine the incidence matrix associated with each of the following graphs:

(a)



(b)



3. For each of the following incidence matrices, construct a graph. Label the vertices $P_1, P_2, \ldots, P_5$.

(a) $A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

4. If

$$\begin{bmatrix} a+b & c+d \\ c-d & a-b \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 10 & 2 \end{bmatrix},$$

find a, b, c, and d.

5. If

$$\begin{bmatrix} a+2b & 2a-b \\ 2c+d & c-2d \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 4 & -3 \end{bmatrix},$$

find a, b, c, and d.

In Exercises 6 through 9, let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix},$$

$$\text{and} \quad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

6. If possible, compute the indicated linear combination:
   (a) $C + E$ and $E + C$ (b) $A + B$
   (c) $D - F$ (d) $-3C + 5O$
   (e) $2C - 3E$ (f) $2B + F$

7. If possible, compute the indicated linear combination:
   (a) $3D + 2F$ (b) $3(2A)$ and $6A$
   (c) $3A + 2A$ and $5A$
   (d) $2(D + F)$ and $2D + 2F$
   (e) $(2 + 3)D$ and $2D + 3D$
   (f) $3(B + D)$

8. If possible, compute the following:
   (a) A^T and $(A^T)^T$
   (b) $(C + E)^T$ and $C^T + E^T$
   (c) $(2D + 3F)^T$ (d) $D - D^T$
   (e) $2A^T + B$ (f) $(3D - 2F)^T$

9. If possible, compute the following:
   (a) $(2A)^T$ (b) $(A - B)^T$
   (c) $(3B^T - 2A)^T$ (d) $(3A^T - 5B^T)^T$
   (e) $(-A)^T$ and $-(A^T)$ (f) $(C + E + F^T)^T$

10. Is the matrix $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$? Justify your answer.

11. Is the matrix $\begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix}$ a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$? Justify your answer.

12. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & -2 & 3 \\ 5 & 2 & 4 \end{bmatrix} \quad \text{and} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If λ is a real number, compute $\lambda I_3 - A$.

13. If A is an $n \times n$ matrix, what are the entries on the main diagonal of $A - A^T$? Justify your answer.

14. Explain why every incidence matrix A associated with a graph is the same as A^T.

15. Let the $n \times n$ matrix A be equal to A^T. Briefly describe the pattern of the entries in A.

16. If $\mathbf{x}$ is an n-vector, show that $\mathbf{x} + \mathbf{0} = \mathbf{x}$.

17. Show that the summation notation satisfies the following properties:
   (a) $\displaystyle\sum_{i=1}^{n} (r_i + s_i)a_i = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} s_i a_i$
   (b) $\displaystyle\sum_{i=1}^{n} c(r_i a_i) = c\left(\sum_{i=1}^{n} r_i a_i\right)$

18. Show that $\displaystyle\sum_{i=1}^{n}\left(\sum_{j=1}^{m} a_{ij}\right) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_{ij}\right)$.

19. Identify the following expressions as true or false. If true, prove the result; if false, give a counterexample.
   (a) $\displaystyle\sum_{i=1}^{n} (a_i + 1) = \left(\sum_{i=1}^{n} a_i\right) + n$
   (b) $\displaystyle\sum_{i=1}^{n}\left(\sum_{j=1}^{m} 1\right) = mn$
   (c) $\displaystyle\sum_{j=1}^{m}\left(\sum_{i=1}^{n} a_i b_j\right) = \left[\sum_{i=1}^{n} a_i\right]\left[\sum_{j=1}^{m} b_j\right]$

20. A large steel manufacturer, who has 2000 employees, lists each employee's salary as a component of a vector $\mathbf{u}$ in R^{2000}. If an 8% across-the-board salary increase has been approved, find an expression involving $\mathbf{u}$ that gives all the new salaries.

21. A brokerage firm records the high and low values of the price of IBM stock each day. The information for a given week is presented in two vectors, $\mathbf{t}$ and $\mathbf{b}$, in R^5, showing the high and low values, respectively. What expression gives the average daily values of the price of IBM stock for the entire 5-day week?

22. For the software you are using, determine the commands to enter a matrix, add matrices, multiply a scalar times a matrix, and obtain the transpose of a matrix for matrices with numerical entries. Practice the commands, using the linear combinations in Example 13.

23. Determine whether the software you are using includes a computer algebra system (CAS), and if it does, do the following:
   (a) Find the command for entering a symbolic matrix. (This command may be different than that for entering a numeric matrix.)
   (b) Enter several symbolic matrices like

$$A = \begin{bmatrix} r & s & t \\ u & v & w \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}.$$

   Compute expressions like $A + B$, $2A$, $3A + B$, $A - 2B$, $A^T + B^T$, etc. (In some systems you must