explicitly indicate scalar multiplication with an asterisk.)

24. For the software you are using, determine whether there is a command that will display a graph for an incidence matrix. If there is, display the graphs for the incidence matrices in Exercise 3 and compare them with those that you drew by hand. Explain why the computer-generated graphs need not be identical to those you drew by hand.

1.3 Matrix Multiplication

In this section we introduce the operation of matrix multiplication. Unlike matrix addition, matrix multiplication has some properties that distinguish it from multiplication of real numbers.

DEFINITION 1.6

The **dot product**, or **inner product**, of the n-vectors in R^n

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{i=1}^{n} a_i b_i .^*$$

The dot product is an important operation that will be used here and in later sections.

EXAMPLE 1

The dot product of

$$\mathbf{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ -2 \\ 1 \end{bmatrix}$$

is

$$\mathbf{u} \cdot \mathbf{v} = (1)(2) + (-2)(3) + (3)(-2) + (4)(1) = -6.$$ ■

EXAMPLE 2

Let $\mathbf{a} = \begin{bmatrix} x \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$. If $\mathbf{a} \cdot \mathbf{b} = -4$, find x.

Solution

We have

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = 4x + 2 + 6 &= -4 \\ 4x + 8 &= -4 \\ x &= -3. \end{aligned}$$ ■

*The dot product of vectors in C^n is defined in Appendix B.2.

EXAMPLE 3

(Computing a Course Average) Suppose that an instructor uses four grades to determine a student's course average: quizzes, two hourly exams, and a final exam. These are weighted as 10%, 30%, 30%, and 30%, respectively. If a student's scores are 78, 84, 62, and 85, respectively, we can compute the course average by letting

$$\mathbf{w} = \begin{bmatrix} 0.10 \\ 0.30 \\ 0.30 \\ 0.30 \end{bmatrix} \quad \text{and} \quad \mathbf{g} = \begin{bmatrix} 78 \\ 84 \\ 62 \\ 85 \end{bmatrix}$$

and computing

$$\mathbf{w} \cdot \mathbf{g} = (0.10)(78) + (0.30)(84) + (0.30)(62) + (0.30)(85) = 77.1.$$

Thus the student's course average is 77.1. ■

■ Matrix Multiplication

DEFINITION 1.7

If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is an $m \times p$ matrix and $B = \begin{bmatrix} b_{ij} \end{bmatrix}$ is a $p \times n$ matrix, then the **product** of A and B, denoted AB, is the $m \times n$ matrix $C = \begin{bmatrix} c_{ij} \end{bmatrix}$, defined by

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} \\ &= \sum_{k=1}^{p} a_{ik}b_{kj} \quad (1 \le i \le m,\ 1 \le j \le n). \end{aligned} \tag{1}$$

Equation (1) says that the i, jth element in the product matrix is the dot product of the transpose of the ith row, $\text{row}_i(A)$—that is, $(\text{row}_i(A))^T$—and the jth column, $\text{col}_j(B)$, of B; this is shown in Figure 1.4.

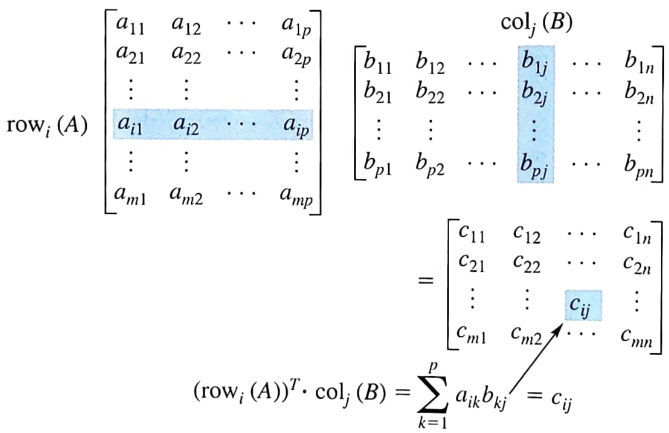

FIGURE 1.4

Observe that the product of A and B is defined only when the number of rows of B is exactly the same as the number of columns of A, as indicated in Figure 1.5.

EXAMPLE 4

Let

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 5 \\ 4 & -3 \\ 2 & 1 \end{bmatrix}.$$

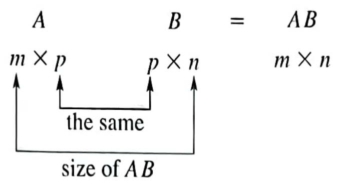

FIGURE 1.5

Then

$$AB = \begin{bmatrix} (1)(-2) + (2)(4) + (-1)(2) & (1)(5) + (2)(-3) + (-1)(1) \\ (3)(-2) + (1)(4) + (4)(2) & (3)(5) + (1)(-3) + (4)(1) \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix}.$$

■

EXAMPLE 5 Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 4 \\ 3 & -1 \\ -2 & 2 \end{bmatrix}.$$

Compute the (3, 2) entry of AB.

Solution

If $AB = C$, then the (3, 2) entry of AB is c_{32}, which is $(\text{row}_3(A))^T \cdot \text{col}_2(B)$. We now have

$$(\text{row}_3(A))^T \cdot \text{col}_2(B) = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix} = -5.$$

■

EXAMPLE 6 Let

$$A = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix}.$$

If $AB = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$, find x and y.

Solution

We have

$$AB = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix} = \begin{bmatrix} 2 + 4x + 3y \\ 4 - 4 + y \end{bmatrix} = \begin{bmatrix} 12 \\ 6 \end{bmatrix}.$$

Then

$$\begin{aligned} 2 + 4x + 3y &= 12 \\ y &= 6, \end{aligned}$$

so $x = -2$ and $y = 6$.

■

The basic properties of matrix multiplication will be considered in the next section. However, multiplication of matrices requires much more care than their addition, since the algebraic properties of matrix multiplication differ from those satisfied by the real numbers. Part of the problem is due to the fact that AB is defined only when the number of columns of A is the same as the number of rows of B. Thus, if A is an $m \times p$ matrix and B is a $p \times n$ matrix, then AB is an $m \times n$ matrix. What about BA? Four different situations may occur:

1. BA may not be defined; this will take place if $n \neq m$.
2. If BA is defined, which means that $m = n$, then BA is $p \times p$ while AB is $m \times m$; thus, if $m \neq p$, AB and BA are of different sizes.
3. If AB and BA are both of the same size, they may be equal.
4. If AB and BA are both of the same size, they may be unequal.

EXAMPLE 7

If A is a 2×3 matrix and B is a 3×4 matrix, then AB is a 2×4 matrix while BA is undefined. ■

EXAMPLE 8

Let A be 2×3 and let B be 3×2. Then AB is 2×2 while BA is 3×3. ■

EXAMPLE 9

Let

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix} \quad \text{while} \quad BA = \begin{bmatrix} 1 & 7 \\ -1 & 3 \end{bmatrix}.$$

Thus $AB \neq BA$. ■

One might ask why matrix equality and matrix addition are defined in such a natural way, while matrix multiplication appears to be much more complicated. Only a thorough understanding of the composition of functions and the relationship that exists between matrices and what are called linear transformations would show that the definition of multiplication given previously is the natural one. These topics are covered later in the book. For now, Example 10 provides a motivation for the definition of matrix multiplication.

EXAMPLE 10

(Ecology) Pesticides are sprayed on plants to eliminate harmful insects. However, some of the pesticide is absorbed by the plant. The pesticides are absorbed by herbivores when they eat the plants that have been sprayed. To determine the amount of pesticide absorbed by a herbivore, we proceed as follows. Suppose that we have three pesticides and four plants. Let a_{ij} denote the amount of pesticide i (in milligrams) that has been absorbed by plant j. This information can be represented by the matrix

$$A = \begin{array}{c} \\ \\ \\ \end{array}\begin{matrix} \text{Plant 1} & \text{Plant 2} & \text{Plant 3} & \text{Plant 4} \\ \end{matrix}$$

$$A = \begin{bmatrix} 2 & 3 & 4 & 3 \\ 3 & 2 & 2 & 5 \\ 4 & 1 & 6 & 4 \end{bmatrix} \begin{matrix} \text{Pesticide 1} \\ \text{Pesticide 2} \\ \text{Pesticide 3} \end{matrix}.$$

Now suppose that we have three herbivores, and let b_{ij} denote the number of plants of type i that a herbivore of type j eats per month. This information can be represented by the matrix

$$B = \begin{array}{c} \begin{array}{ccc} \text{Herbivore 1} & \text{Herbivore 2} & \text{Herbivore 3} \end{array} \\ \left[\begin{array}{ccc} 20 & 12 & 8 \\ 28 & 15 & 15 \\ 30 & 12 & 10 \\ 40 & 16 & 20 \end{array}\right] \end{array} \begin{array}{l} \text{Plant 1} \\ \text{Plant 2} \\ \text{Plant 3} \\ \text{Plant 4} \end{array}.$$

The (i, j) entry in AB gives the amount of pesticide of type i that animal j has absorbed. Thus, if $i = 2$ and $j = 3$, the (2, 3) entry in AB is

$$\begin{aligned}(\text{row}_2(A))^T \cdot \text{col}_3(B) &= 3(8) + 2(15) + 2(10) + 5(20) \\ &= 174 \text{ mg of pesticide 2 absorbed by herbivore 3.}\end{aligned}$$

If we now have p carnivores (such as a human) who eat the herbivores, we can repeat the analysis to find out how much of each pesticide has been absorbed by each carnivore. ■

It is sometimes useful to be able to find a column in the matrix product AB without having to multiply the two matrices. It is not difficult to show (Exercise 46) that the jth column of the matrix product AB is equal to the matrix product $A\text{col}_j(B)$.

EXAMPLE 11 Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}.$$

Then the second column of AB is

$$A\text{col}_2(B) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \\ 7 \end{bmatrix}.$$

■

Remark If $\mathbf{u}$ and $\mathbf{v}$ are n-vectors ($n \times 1$ matrices), then it is easy to show by matrix multiplication (Exercise 41) that

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}.$$

This observation is applied in Chapter 5.

■ The Matrix–Vector Product Written in Terms of Columns

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

be an $m \times n$ matrix and let

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

be an n-vector, that is, an $n \times 1$ matrix. Since A is $m \times n$ and $\mathbf{c}$ is $n \times 1$, the matrix product $A\mathbf{c}$ is the $m \times 1$ matrix

$$A\mathbf{c} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} (\text{row}_1(A))^T \cdot \mathbf{c} \\ (\text{row}_2(A))^T \cdot \mathbf{c} \\ \vdots \\ (\text{row}_m(A))^T \cdot \mathbf{c} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1n}c_n \\ a_{21}c_1 + a_{22}c_2 + \cdots + a_{2n}c_n \\ \vdots \\ a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mn}c_n \end{bmatrix}. \qquad (2)$$

This last expression can be written as

$$c_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + c_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \qquad (3)$$

$$= c_1\text{col}_1(A) + c_2\text{col}_2(A) + \cdots + c_n\text{col}_n(A).$$

Thus the product $A\mathbf{c}$ of an $m \times n$ matrix A and an $n \times 1$ matrix $\mathbf{c}$ can be written as a linear combination of the columns of A, where the coefficients are the entries in the matrix $\mathbf{c}$.

In our study of linear systems of equations we shall see that these systems can be expressed in terms of a matrix–vector product. This point of view provides us with an important way to think about solutions of linear systems.

EXAMPLE 12

Let

$$A = \begin{bmatrix} 2 & -1 & -3 \\ 4 & 2 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}.$$

Then the product $A\mathbf{c}$, written as a linear combination of the columns of A, is

$$A\mathbf{c} = \begin{bmatrix} 2 & -1 & -3 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 2 \\ 4 \end{bmatrix} - 3\begin{bmatrix} -1 \\ 2 \end{bmatrix} + 4\begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ -6 \end{bmatrix}.$$ ■

If A is an $m \times p$ matrix and B is a $p \times n$ matrix, we can then conclude that the jth column of the product AB can be written as a linear combination of the columns of matrix A, where the coefficients are the entries in the jth column of matrix B:

$$\text{col}_j(AB) = A\text{col}_j(B) = b_{1j}\text{col}_1(A) + b_{2j}\text{col}_2(A) + \cdots + b_{pj}\text{col}_p(A).$$

EXAMPLE 13 If A and B are the matrices defined in Example 11, then

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 7 & 6 \\ 6 & 17 & 16 \\ 17 & 7 & 1 \end{bmatrix}.$$

The columns of AB as linear combinations of the columns of A are given by

$$\text{col}_1(AB) = \begin{bmatrix} 4 \\ 6 \\ 17 \end{bmatrix} = A\text{col}_1(B) = -2\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\text{col}_2(AB) = \begin{bmatrix} 7 \\ 17 \\ 7 \end{bmatrix} = A\text{col}_2(B) = 3\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\text{col}_3(AB) = \begin{bmatrix} 6 \\ 16 \\ 1 \end{bmatrix} = A\text{col}_3(B) = 4\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}.$$

■

■ Linear Systems

Consider the linear system of m equations in n unknowns,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{4}$$

Now define the following matrices:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Then

$$A\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}. \quad (5)$$

The entries in the product $A\mathbf{x}$ at the end of (5) are merely the left sides of the equations in (4). Hence the linear system (4) can be written in matrix form as

$$A\mathbf{x} = \mathbf{b}.$$

The matrix A is called the **coefficient matrix** of the linear system (4), and the matrix

$$\left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right],$$

obtained by adjoining column $\mathbf{b}$ to A, is called the **augmented matrix** of the linear system (4). The augmented matrix of (4) is written as $\left[A \;\vdots\; \mathbf{b}\right]$. Conversely, any matrix with more than one column can be thought of as the augmented matrix of a linear system. The coefficient and augmented matrices play key roles in our method for solving linear systems.

Recall from Section 1.1 that if

$$b_1 = b_2 = \cdots = b_m = 0$$

in (4), the linear system is called a **homogeneous system**. A homogeneous system can be written as

$$A\mathbf{x} = \mathbf{0},$$

where A is the coefficient matrix.

EXAMPLE 14 Consider the linear system

$$\begin{aligned} -2x \qquad + z &= 5 \\ 2x + 3y - 4z &= 7 \\ 3x + 2y + 2z &= 3. \end{aligned}$$

Letting

$$A = \begin{bmatrix} -2 & 0 & 1 \\ 2 & 3 & -4 \\ 3 & 2 & 2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix},$$

we can write the given linear system in matrix form as

$$A\mathbf{x} = \mathbf{b}.$$

The coefficient matrix is A, and the augmented matrix is

$$\left[\begin{array}{rrr|r} -2 & 0 & 1 & 5 \\ 2 & 3 & -4 & 7 \\ 3 & 2 & 2 & 3 \end{array}\right].$$

■

EXAMPLE 15 The matrix

$$\left[\begin{array}{rrr|r} 2 & -1 & 3 & 4 \\ 3 & 0 & 2 & 5 \end{array}\right]$$

is the augmented matrix of the linear system

$$\begin{aligned} 2x - y + 3z &= 4 \\ 3x \quad\;\; + 2z &= 5. \end{aligned}$$

■

We can express (5) in another form, as follows, using (2) and (3):

$$A\mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \\ \vdots \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \\ \vdots \\ a_{m2}x_2 \end{bmatrix} + \cdots + \begin{bmatrix} a_{1n}x_n \\ a_{2n}x_n \\ \vdots \\ a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$= x_1\,\mathbf{col}_1(A) + x_2\,\mathbf{col}_2(A) + \cdots + x_n\,\mathbf{col}_n(A).$$

Thus $A\mathbf{x}$ is a linear combination of the columns of A with coefficients that are the entries of $\mathbf{x}$. It follows that the matrix form of a linear system, $A\mathbf{x} = \mathbf{b}$, can be expressed as

$$x_1\,\mathbf{col}_1(A) + x_2\,\mathbf{col}_2(A) + \cdots + x_n\,\mathbf{col}_n(A) = \mathbf{b}. \tag{6}$$

Conversely, an equation of the form in (6) always describes a linear system of the form in (4).

EXAMPLE 16 Consider the linear system $A\mathbf{x} = \mathbf{b}$, where the coefficient matrix

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 4 & -5 & 6 \\ 0 & 7 & -3 \\ -1 & 2 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

Writing $A\mathbf{x} = \mathbf{b}$ as a linear combination of the columns of A as in (6), we have

$$x_1 \begin{bmatrix} 3 \\ 4 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -5 \\ 7 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 6 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

The expression for the linear system $A\mathbf{x} = \mathbf{b}$ as shown in (6), provides an important way to think about solutions of linear systems.

$A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ can be expressed as a linear combination of the columns of the matrix A.

We encounter this approach in Chapter 2.

Key Terms

Dot product (inner product)
Matrix–vector product
Coefficient matrix
Augmented matrix

1.3 Exercises

In Exercises 1 and 2, compute $\mathbf{a} \cdot \mathbf{b}$.

1. (a) $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

(c) $\mathbf{a} = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

(d) $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

2. (a) $\mathbf{a} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(b) $\mathbf{a} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(c) $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

(d) $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

3. Let $\mathbf{a} = \mathbf{b} = \begin{bmatrix} -3 \\ 2 \\ x \end{bmatrix}$. If $\mathbf{a} \cdot \mathbf{b} = 17$, find x.

4. Determine the value of x so that $\mathbf{v} \cdot \mathbf{w} = 0$, where

$$\mathbf{v} = \begin{bmatrix} 1 \\ -3 \\ 4 \\ x \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} x \\ 2 \\ -1 \\ 1 \end{bmatrix}.$$

5. Determine values of x and y so that $\mathbf{v} \cdot \mathbf{w} = 0$ and $\mathbf{v} \cdot \mathbf{u} = 0$, where $\mathbf{v} = \begin{bmatrix} x \\ 1 \\ y \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}$.

6. Determine values of x and y so that $\mathbf{v} \cdot \mathbf{w} = 0$ and $\mathbf{v} \cdot \mathbf{u} = 0$, where $\mathbf{v} = \begin{bmatrix} x \\ 1 \\ y \end{bmatrix}$, $\mathbf{w} = \begin{bmatrix} x \\ -2 \\ 0 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 0 \\ -9 \\ y \end{bmatrix}$.

7. Let $\mathbf{w} = \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix}$. Compute $\mathbf{w} \cdot \mathbf{w}$.

8. Find all values of x so that $\mathbf{u} \cdot \mathbf{u} = 50$, where $\mathbf{u} = \begin{bmatrix} x \\ 3 \\ 4 \end{bmatrix}$.

9. Find all values of x so that $\mathbf{v} \cdot \mathbf{v} = 1$, where $\mathbf{v} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ x \end{bmatrix}$.

10. Let $A = \begin{bmatrix} 1 & 2 & x \\ 3 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} y \\ x \\ 1 \end{bmatrix}$.

If $AB = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$, find x and y.

Consider the following matrices for Exercises 11 through 15:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix},$$

$$E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}, \quad \textit{and} \quad F = \begin{bmatrix} -1 & 2 \\ 0 & 4 \\ 3 & 5 \end{bmatrix}.$$

11. If possible, compute the following:
(a) AB (b) BA (c) $F^T E$
(d) $CB + D$ (e) $AB + D^2$, where $D^2 = DD$

12. If possible, compute the following:
(a) $DA + B$ (b) EC (c) CE
(d) $EB + F$ (e) $FC + D$

13. If possible, compute the following:
(a) $FD - 3B$ (b) $AB - 2D$
(c) $F^T B + D$ (d) $2F - 3(AE)$
(e) $BD + AE$

14. If possible, compute the following:
(a) $A(BD)$ (b) $(AB)D$
(c) $A(C + E)$ (d) $AC + AE$
(e) $(2AB)^T$ and $2(AB)^T$ (f) $A(C - 3E)$

15. If possible, compute the following:
(a) A^T (b) $(A^T)^T$
(c) $(AB)^T$ (d) $B^T A^T$
(e) $(C + E)^T B$ and $C^T B + E^T B$
(f) $A(2B)$ and $2(AB)$

16. Let $A = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 4 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} -3 & 0 & 1 \end{bmatrix}$. If possible, compute the following:
(a) AB^T (b) CA^T (c) $(BA^T)C$
(d) $A^T B$ (e) CC^T (f) $C^T C$
(g) $B^T CAA^T$

17. Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$.

Compute the following entries of AB:
(a) the (1, 2) entry (b) the (2, 3) entry
(c) the (3, 1) entry (d) the (3, 3) entry

18. If $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$, compute DI_2 and $I_2 D$.

19. Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}.$$

Show that $AB \neq BA$.

20. If A is the matrix in Example 4 and O is the 3×2 matrix every one of whose entries is zero, compute AO.

In Exercises 21 and 22, let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 4 & -2 & 3 \\ 2 & 1 & 5 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 3 & 3 & -3 & 4 \\ 4 & 2 & 5 & 1 \end{bmatrix}.$$

21. Using the method in Example 11, compute the following columns of AB:
(a) the first column (b) the third column

22. Using the method in Example 11, compute the following columns of AB:
(a) the second column (b) the fourth column

23. Let

$$A = \begin{bmatrix} 2 & -3 & 4 \\ -1 & 2 & 3 \\ 5 & -1 & -2 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

Express $A\mathbf{c}$ as a linear combination of the columns of A.

24. Let

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 4 & 3 \\ 3 & 0 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 2 & 4 \end{bmatrix}.$$

Express the columns of AB as linear combinations of the columns of A.

25. Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$.

(a) Verify that $AB = 3\mathbf{a}_1 + 5\mathbf{a}_2 + 2\mathbf{a}_3$, where $\mathbf{a}_j$ is the jth column of A for $j = 1, 2, 3$.

(b) Verify that $AB = \begin{bmatrix} (\text{row}_1(A))B \\ (\text{row}_2(A))B \end{bmatrix}$.

26. (a) Find a value of r so that $AB^T = 0$, where $A = \begin{bmatrix} r & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -1 \end{bmatrix}$.

(b) Give an alternative way to write this product.

27. Find a value of r and a value of s so that $AB^T = 0$, where $A = \begin{bmatrix} 1 & r & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 2 & s \end{bmatrix}$.

28. (a) Let A be an $m \times n$ matrix with a row consisting entirely of zeros. Show that if B is an $n \times p$ matrix, then AB has a row of zeros.

(b) Let A be an $m \times n$ matrix with a column consisting entirely of zeros and let B be $p \times m$. Show that BA has a column of zeros.

29. Let $A = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 5 & 0 \end{bmatrix}$ with $\mathbf{a}_j$ = the jth column of A, $j = 1, 2, 3$. Verify that

$$A^T A = \begin{bmatrix} \mathbf{a}_1^T\mathbf{a}_1 & \mathbf{a}_1^T\mathbf{a}_2 & \mathbf{a}_1^T\mathbf{a}_3 \\ \mathbf{a}_2^T\mathbf{a}_1 & \mathbf{a}_2^T\mathbf{a}_2 & \mathbf{a}_2^T\mathbf{a}_3 \\ \mathbf{a}_3^T\mathbf{a}_1 & \mathbf{a}_3^T\mathbf{a}_2 & \mathbf{a}_3^T\mathbf{a}_3 \end{bmatrix}.$$

30. Consider the following linear system:

$$\begin{aligned} 2x_1 + 3x_2 - 3x_3 + x_4 + x_5 &= 7 \\ 3x_1 \qquad + 2x_3 \qquad + 3x_5 &= -2 \\ 2x_1 + 3x_2 \qquad - 4x_4 \qquad &= 3 \\ x_3 + x_4 + x_5 &= 5. \end{aligned}$$

(a) Find the coefficient matrix.

(b) Write the linear system in matrix form.

(c) Find the augmented matrix.

31. Write the linear system whose augmented matrix is

$$\left[\begin{array}{cccc|c} -2 & -1 & 0 & 4 & 5 \\ -3 & 2 & 7 & 8 & 3 \\ 1 & 0 & 0 & 2 & 4 \\ 3 & 0 & 1 & 3 & 6 \end{array}\right].$$

32. Write the following linear system in matrix form:

$$\begin{aligned} -2x_1 + 3x_2 &= 5 \\ x_1 - 5x_2 &= 4 \end{aligned}$$

33. Write the following linear system in matrix form:

$$\begin{aligned} 2x_1 + 3x_2 &= 0 \\ 3x_2 + x_3 &= 0 \\ 2x_1 - x_2 &= 0 \end{aligned}$$

34. Write the linear system whose augmented matrix is

(a) $\left[\begin{array}{cccc|c} 2 & 1 & 3 & 4 & 0 \\ 3 & -1 & 2 & 0 & 3 \\ -2 & 1 & -4 & 3 & 2 \end{array}\right]$.

(b) $\left[\begin{array}{cccc|c} 2 & 1 & 3 & 4 & 0 \\ 3 & -1 & 2 & 0 & 3 \\ -2 & 1 & -4 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$.

35. How are the linear systems obtained in Exercise 34 related?

36. Write each of the following linear systems as a linear combination of the columns of the coefficient matrix:

(a) $\begin{aligned} 3x_1 + 2x_2 + x_3 &= 4 \\ x_1 - x_2 + 4x_3 &= -2 \end{aligned}$

(b) $\begin{aligned} -x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= -2 \\ 3x_1 + x_2 &= 1 \end{aligned}$

37. Write each of the following linear combinations of columns as a linear system of the form in (4):

(a) $x_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

(b) $x_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

38. Write each of the following as a linear system in matrix form:

(a) $x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

39. Determine a solution to each of the following linear systems, using the fact that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is a linear combination of the columns of A:

(a) $\begin{bmatrix} 1 & 2 & 1 \\ -3 & 6 & -3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \end{bmatrix}$

40. Construct a coefficient matrix A so that $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is a solution to the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Can there be more than one such coefficient matrix? Explain.

41. Show that if $\mathbf{u}$ and $\mathbf{v}$ are n-vectors, then $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

42. Let A be an $m \times n$ matrix and B an $n \times p$ matrix. What, if anything, can you say about the matrix product AB when

(a) A has a column consisting entirely of zeros?

(b) B has a row consisting entirely of zeros?

43. If $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is an $n \times n$ matrix, then the **trace** of A, Tr(A), is defined as the sum of all elements on the main diagonal of A, $\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$. Show each of the following:

(a) $\mathrm{Tr}(cA) = c\,\mathrm{Tr}(A)$, where c is a real number

(b) $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$

(c) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

(d) $\mathrm{Tr}(A^T) = \mathrm{Tr}(A)$

(e) $\mathrm{Tr}(A^T A) \geq 0$

44. Compute the trace (see Exercise 43) of each of the following matrices:

(a) $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 2 & 2 & 3 \\ 2 & 4 & 4 \\ 3 & -2 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

45. Show that there are no 2×2 matrices A and B such that $AB - BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

46. **(a)** Show that the jth column of the matrix product AB is equal to the matrix product $A\mathbf{b}_j$, where $\mathbf{b}_j$ is the jth column of B. It follows that the product AB can be written in terms of columns as

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{bmatrix}.$$

(b) Show that the ith row of the matrix product AB is equal to the matrix product $\mathbf{a}_i B$, where $\mathbf{a}_i$ is the ith row of A. It follows that the product AB can be written in terms of rows as

$$AB = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix}.$$

47. Show that the jth column of the matrix product AB is a linear combination of the columns of A with coefficients the entries in $\mathbf{b}_j$, the jth column of B.

48. The vector

$$\mathbf{u} = \begin{bmatrix} 20 \\ 30 \\ 80 \\ 10 \end{bmatrix}$$

gives the number of receivers, CD players, speakers, and DVD recorders that are on hand in an audio shop. The vector

$$\mathbf{v} = \begin{bmatrix} 200 \\ 120 \\ 80 \\ 70 \end{bmatrix}$$

gives the price (in dollars) of each receiver, CD player, speaker, and DVD recorder, respectively. What does the dot product $\mathbf{u} \cdot \mathbf{v}$ tell the shop owner?

49. ***(Manufacturing Costs)*** A furniture manufacturer makes chairs and tables, each of which must go through an assembly process and a finishing process. The times required for these processes are given (in hours) by the matrix

$$A = \begin{array}{c} \text{Assembly process} \quad \text{Finishing process} \\ \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} \begin{array}{l} \text{Chair} \\ \text{Table} \end{array} \end{array}.$$

The manufacturer has a plant in Salt Lake City and another in Chicago. The hourly rates for each of the processes are given (in dollars) by the matrix

$$B = \begin{array}{c} \text{Salt Lake City} \quad \text{Chicago} \\ \begin{bmatrix} 9 & 10 \\ 10 & 12 \end{bmatrix} \begin{array}{l} \text{Assembly process} \\ \text{Finishing process} \end{array} \end{array}.$$

What do the entries in the matrix product AB tell the manufacturer?

50. ***(Medicine)*** A diet research project includes adults and children of both sexes. The composition of the participants in the project is given by the matrix

$$A = \begin{array}{c} \text{Adults} \quad \text{Children} \\ \begin{bmatrix} 80 & 120 \\ 100 & 200 \end{bmatrix} \begin{array}{l} \text{Male} \\ \text{Female} \end{array} \end{array}.$$

The number of daily grams of protein, fat, and carbohydrate consumed by each child and adult is given by the matrix

$$B = \begin{array}{c} \text{Protein} \quad \text{Fat} \quad \text{Carbohydrate} \\ \begin{bmatrix} 20 & 20 & 20 \\ 10 & 20 & 30 \end{bmatrix} \begin{array}{l} \text{Adult} \\ \text{Child} \end{array} \end{array}.$$