(a) How many grams of protein are consumed daily by the males in the project?

(b) How many grams of fat are consumed daily by the females in the project?

51. Let $\mathbf{x}$ be an n-vector.

(a) Is it possible for $\mathbf{x} \cdot \mathbf{x}$ to be negative? Explain.

(b) If $\mathbf{x} \cdot \mathbf{x} = 0$, what is $\mathbf{x}$?

52. Let $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ be n-vectors and let k be a real number.

(a) Show that $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$.

(b) Show that $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$.

(c) Show that $(k\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (k\mathbf{b}) = k(\mathbf{a} \cdot \mathbf{b})$.

53. Let A be an $m \times n$ matrix whose entries are real numbers. Show that if $AA^T = O$ (the $m \times m$ matrix all of whose entries are zero), then $A = O$.

54. Use the matrices A and C in Exercise 11 and the matrix multiplication command in your software to compute AC and CA. Discuss the results.

55. Using your software, compute $B^T B$ and BB^T for

$$B = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \end{bmatrix}.$$

Discuss the nature of the results.

1.4 Algebraic Properties of Matrix Operations

In this section we consider the algebraic properties of the matrix operations just defined. Many of these properties are similar to the familiar properties that hold for real numbers. However, there will be striking differences between the set of real numbers and the set of matrices in their algebraic behavior under certain operations—for example, under multiplication (as seen in Section 1.3). The proofs of most of the properties will be left as exercises.

Theorem 1.1 **Properties of Matrix Addition**

Let A, B, and C be $m \times n$ matrices.

(a) $A + B = B + A$.

(b) $A + (B + C) = (A + B) + C$.

(c) There is a unique $m \times n$ matrix O such that

$$A + O = A \qquad (1)$$

for any $m \times n$ matrix A. The matrix O is called the $m \times n$ **zero matrix**.

(d) For each $m \times n$ matrix A, there is a unique $m \times n$ matrix D such that

$$A + D = O. \qquad (2)$$

We shall write D as $-A$, so (2) can be written as

$$A + (-A) = O.$$

The matrix $-A$ is called the **negative** of A. We also note that $-A$ is $(-1)A$.

Proof

(a) Let

$$A = \begin{bmatrix} a_{ij} \end{bmatrix}, \quad B = \begin{bmatrix} b_{ij} \end{bmatrix},$$

$$A + B = C = \begin{bmatrix} c_{ij} \end{bmatrix}, \quad \text{and} \quad B + A = D = \begin{bmatrix} d_{ij} \end{bmatrix}.$$

We must show that $c_{ij} = d_{ij}$ for all i, j. Now $c_{ij} = a_{ij} + b_{ij}$ and $d_{ij} = b_{ij} + a_{ij}$ for all i, j. Since a_{ij} and b_{ij} are real numbers, we have $a_{ij} + b_{ij} = b_{ij} + a_{ij}$, which implies that $c_{ij} = d_{ij}$ for all i, j.

(c) Let $U = [u_{ij}]$. Then $A + U = A$ if and only if[†] $a_{ij} + u_{ij} = a_{ij}$, which holds if and only if $u_{ij} = 0$. Thus U is the $m \times n$ matrix all of whose entries are zero: U is denoted by O. ■

The 2×2 zero matrix is

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

EXAMPLE 1

If

$$A = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix},$$

then

$$\begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4+0 & -1+0 \\ 2+0 & 3+0 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}.$$ ■

The 2×3 zero matrix is

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

EXAMPLE 2

If $A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & 4 & 3 \end{bmatrix}$, then $-A = \begin{bmatrix} -1 & -3 & 2 \\ 2 & -4 & -3 \end{bmatrix}$. ■

Theorem 1.2 **Properties of Matrix Multiplication**

(a) If A, B, and C are matrices of the appropriate sizes, then

$$A(BC) = (AB)C.$$

(b) If A, B, and C are matrices of the appropriate sizes, then

$$(A + B)C = AC + BC.$$

(c) If A, B, and C are matrices of the appropriate sizes, then

$$C(A + B) = CA + CB. \qquad (3)$$

Proof

(a) Suppose that A is $m \times n$, B is $n \times p$, and C is $p \times q$. We shall prove the result for the special case $m = 2$, $n = 3$, $p = 4$, and $q = 3$. The general proof is completely analogous.

Let $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$, $AB = D = [d_{ij}]$, $BC = E = [e_{ij}]$, $(AB)C = F = [f_{ij}]$, and $A(BC) = G = [g_{ij}]$. We must show that $f_{ij} = g_{ij}$ for all i, j. Now

$$f_{ij} = \sum_{k=1}^{4} d_{ik} c_{kj} = \sum_{k=1}^{4} \left(\sum_{r=1}^{3} a_{ir} b_{rk} \right) c_{kj}$$

[†]The connector "if and only if" means that both statements are true or both statements are false. Thus (i) if $A + U = A$, then $a_{ij} + u_{ij} = a_{ij}$; and (ii) if $a_{ij} + u_{ij} = a_{ij}$, then $A + U = A$. See Appendix C, "Introduction to Proofs."

and

$$g_{ij} = \sum_{r=1}^{3} a_{ir} e_{rj} = \sum_{r=1}^{3} a_{ir} \left(\sum_{k=1}^{4} b_{rk} c_{kj} \right).$$

Then, by the properties satisfied by the summation notation,

$$\begin{aligned} f_{ij} &= \sum_{k=1}^{4} (a_{i1} b_{1k} + a_{i2} b_{2k} + a_{i3} b_{3k}) c_{kj} \\ &= a_{i1} \sum_{k=1}^{4} b_{1k} c_{kj} + a_{i2} \sum_{k=1}^{4} b_{2k} c_{kj} + a_{i3} \sum_{k=1}^{4} b_{3k} c_{kj} \\ &= \sum_{r=1}^{3} a_{ir} \left(\sum_{k=1}^{4} b_{rk} c_{kj} \right) = g_{ij}. \end{aligned}$$

The proofs of (b) and (c) are left as Exercise 4. ■

EXAMPLE 3 Let

$$A = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & -1 & 1 & 0 \\ 0 & 2 & 2 & 2 \\ 3 & 0 & -1 & 3 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix}.$$

Then

$$A(BC) = \begin{bmatrix} 5 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 3 & 7 \\ 8 & -4 & 6 \\ 9 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 43 & 16 & 56 \\ 12 & 30 & 8 \end{bmatrix}$$

and

$$(AB)C = \begin{bmatrix} 19 & -1 & 6 & 13 \\ 16 & -8 & -8 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & -3 & 0 \\ 0 & 0 & 3 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 43 & 16 & 56 \\ 12 & 30 & 8 \end{bmatrix}.$$ ■

EXAMPLE 4 Let

$$A = \begin{bmatrix} 2 & 2 & 3 \\ 3 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & -1 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}.$$

Then

$$(A + B)C = \begin{bmatrix} 2 & 2 & 4 \\ 5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 18 & 0 \\ 12 & 3 \end{bmatrix}$$

and (verify)

$$AC + BC = \begin{bmatrix} 15 & 1 \\ 7 & -4 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 18 & 0 \\ 12 & 3 \end{bmatrix}.$$

■

Recall Example 9 in Section 1.3, which shows that AB need not always equal BA. This is the first significant difference between multiplication of matrices and multiplication of real numbers.

Theorem 1.3 **Properties of Scalar Multiplication**

If r and s are real numbers and A and B are matrices of the appropriate sizes, then

(a) $r(sA) = (rs)A$

(b) $(r + s)A = rA + sA$

(c) $r(A + B) = rA + rB$

(d) $A(rB) = r(AB) = (rA)B$

Proof

Exercises 13, 14, 16, and 18. ■

EXAMPLE 5

Let

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Then

$$2(3A) = 2\begin{bmatrix} 12 & 6 & 9 \\ 6 & -9 & 12 \end{bmatrix} = \begin{bmatrix} 24 & 12 & 18 \\ 12 & -18 & 24 \end{bmatrix} = 6A.$$

We also have

$$A(2B) = \begin{bmatrix} 4 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} 6 & -4 & 2 \\ 4 & 0 & -2 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 32 & -10 & 16 \\ 0 & 0 & 26 \end{bmatrix} = 2(AB).$$

■

EXAMPLE 6

Scalar multiplication can be used to change the size of entries in a matrix to meet prescribed properties. Let

$$A = \begin{bmatrix} 3 \\ 7 \\ 2 \\ 1 \end{bmatrix}.$$

Then for $k = \frac{1}{7}$, the largest entry of kA is 1. Also if the entries of A represent the volume of products in gallons, for $k = 4$, kA gives the volume in quarts. ■

So far we have seen that multiplication and addition of matrices have much in common with multiplication and addition of real numbers. We now look at some properties of the transpose.

Theorem 1.4 **Properties of Transpose**

If r is a scalar and A and B are matrices of the appropriate sizes, then

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(AB)^T = B^T A^T$

(d) $(rA)^T = rA^T$

Proof

We leave the proofs of (a), (b), and (d) as Exercises 26 and 27.

(c) Let $A = \left[a_{ij}\right]$ and $B = \left[b_{ij}\right]$; let $AB = C = \left[c_{ij}\right]$. We must prove that c^T_{ij} is the (i, j) entry in $B^T A^T$. Now

$$c^T_{ij} = c_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki} = \sum_{k=1}^{n} a^T_{kj} b^T_{ik}$$

$$= \sum_{k=1}^{n} b^T_{ik} a^T_{kj} = \text{the } (i, j) \text{ entry in } B^T A^T.$$

EXAMPLE 7 Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}.$$

Then

$$A^T = \begin{bmatrix} 1 & -2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 3 & 3 \\ -1 & 2 \\ 2 & -1 \end{bmatrix}.$$

Also,

$$A + B = \begin{bmatrix} 4 & 1 & 5 \\ 1 & 2 & 0 \end{bmatrix} \quad \text{and} \quad (A + B)^T = \begin{bmatrix} 4 & 1 \\ 1 & 2 \\ 5 & 0 \end{bmatrix}.$$

Now

$$A^T + B^T = \begin{bmatrix} 4 & 1 \\ 1 & 2 \\ 5 & 0 \end{bmatrix} = (A + B)^T.$$

EXAMPLE 8 Let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 12 & 5 \\ 7 & -3 \end{bmatrix} \quad \text{and} \quad (AB)^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}.$$

On the other hand,

$$A^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix}.$$

Then

$$B^T A^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix} = (AB)^T.$$

■

We also note two other peculiarities of matrix multiplication. If a and b are real numbers, then $ab = 0$ can hold only if a or b is zero. However, this is not true for matrices.

EXAMPLE 9

If

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix},$$

then neither A nor B is the zero matrix, but $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. ■

If a, b, and c are real numbers for which $ab = ac$ and $a \neq 0$, it follows that $b = c$. That is, we can cancel out the nonzero factor a. However, the cancellation law does not hold for matrices, as the following example shows.

EXAMPLE 10

If

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix},$$

then

$$AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix},$$

but $B \neq C$. ■

We summarize some of the differences between matrix multiplication and the multiplication of real numbers as follows: For matrices A, B, and C of the appropriate sizes,

1. AB need not equal BA.
2. AB may be the zero matrix with $A \neq O$ and $B \neq O$.
3. AB may equal AC with $B \neq C$.

In this section we have developed a number of properties about matrices and their transposes. If a future problem involves these concepts, refer to these properties to help solve the problem. These results can be used to develop many more results.

Key Terms

Properties of matrix addition
Zero matrix
Properties of matrix multiplication
Properties of scalar multiplication
Properties of transpose

1.4 Exercises

1. Prove Theorem 1.1(b).
2. Prove Theorem 1.1(d).
3. Verify Theorem 1.2(a) for the following matrices:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & 4 \end{bmatrix},$$

$$\text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 1 & 2 \end{bmatrix}.$$

4. Prove Theorem 1.2(b) and (c).
5. Verify Theorem 1.2(c) for the following matrices:

$$A = \begin{bmatrix} 2 & -3 & 2 \\ 3 & -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & -2 \end{bmatrix},$$

$$\text{and} \quad C = \begin{bmatrix} 1 & -3 \\ -3 & 4 \end{bmatrix}.$$

6. Let $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ be the $n \times n$ matrix defined by $a_{ii} = k$ and $a_{ij} = 0$ if $i \neq j$. Show that if B is any $n \times n$ matrix, then $AB = kB$.
7. Let A be an $m \times n$ matrix and $C = \begin{bmatrix} c_1 & c_2 & \cdots & c_m \end{bmatrix}$ a $1 \times m$ matrix. Prove that

$$CA = \sum_{j=1}^{m} c_j A_j,$$

where A_j is the jth row of A.

8. Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.
   (a) Determine a simple expression for A^2.
   (b) Determine a simple expression for A^3.
   (c) Conjecture the form of a simple expression for A^k, k a positive integer.
   (d) Prove or disprove your conjecture in part (c).
9. Find a pair of unequal 2×2 matrices A and B, other than those given in Example 9, such that $AB = O$.
10. Find two different 2×2 matrices A such that $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
11. Find two unequal 2×2 matrices A and B such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
12. Find two different 2×2 matrices A such that $A^2 = O$.
13. Prove Theorem 1.3(a).
14. Prove Theorem 1.3(b).
15. Verify Theorem 1.3(b) for $r = 4$, $s = -2$, and $A = \begin{bmatrix} 2 & -3 \\ 4 & 2 \end{bmatrix}$.
16. Prove Theorem 1.3(c).
17. Verify Theorem 1.3(c) for $r = -3$,

$$A = \begin{bmatrix} 4 & 2 \\ 1 & -3 \\ 3 & 2 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 & 2 \\ 4 & 3 \\ -2 & 1 \end{bmatrix}.$$

18. Prove Theorem 1.3(d).
19. Verify Theorem 1.3(d) for the following matrices:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 & 2 \\ 1 & -3 & 4 \end{bmatrix},$$

$$\text{and} \quad r = -3.$$

20. The matrix A contains the weight (in pounds) of objects packed on board a spacecraft on earth. The objects are to be used on the moon where things weigh about $\frac{1}{6}$ as much. Write an expression kA that calculates the weight of the objects on the moon.
21. (a) A is a 360×2 matrix. The first column of A is $\cos 0°, \cos 1°, \ldots, \cos 359°$; and the second column is $\sin 0°, \sin 1°, \ldots, \sin 359°$. The graph of the ordered pairs in A is a circle of radius 1 centered at the origin. Write an expression kA for ordered pairs whose graph is a circle of radius 3 centered at the origin.
   (b) Explain how to prove the claims about the circles in part (a).
22. Determine a scalar r such that $A\mathbf{x} = r\mathbf{x}$, where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

23. Determine a scalar r such that $A\mathbf{x} = r\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \\ 1 \end{bmatrix}.$$

24. Prove that if $A\mathbf{x} = r\mathbf{x}$ for $n \times n$ matrix A, $n \times 1$ matrix $\mathbf{x}$, and scalar r, then $A\mathbf{y} = r\mathbf{y}$, where $\mathbf{y} = s\mathbf{x}$ for any scalar s.

25. Determine a scalar s such that $A^2\mathbf{x} = s\mathbf{x}$ when $A\mathbf{x} = r\mathbf{x}$.

26. Prove Theorem 1.4(a).

27. Prove Theorem 1.4(b) and (d).

28. Verify Theorem 1.4(a), (b), and (d) for

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 & -1 \\ -2 & 1 & 5 \end{bmatrix},$$

and $r = -4$.

29. Verify Theorem 1.4(c) for

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}.$$

30. Let

$$A = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix}.$$

(a) Compute $(AB^T)C$.

(b) Compute B^TC and multiply the result by A on the right. (*Hint*: B^TC is 1×1).

(c) Explain why $(AB^T)C = (B^TC)A$.

31. Determine a constant k such that $(kA)^T(kA) = 1$, where $A = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$. Is there more than one value of k that could be used?

32. Find three 2×2 matrices, A, B, and C such that $AB = AC$ with $B \neq C$ and $A \neq O$.

33. Let A be an $n \times n$ matrix and c a real number. Show that if $cA = O$, then $c = 0$ or $A = O$.

34. Determine all 2×2 matrices A such that $AB = BA$ for any 2×2 matrix B.

35. Show that $(A - B)^T = A^T - B^T$.

36. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$.

(a) Show that $\mathbf{x}_1 + \mathbf{x}_2$ is a solution.

(b) Show that $\mathbf{x}_1 - \mathbf{x}_2$ is a solution.

(c) For any scalar r, show that $r\mathbf{x}_1$ is a solution.

(d) For any scalars r and s, show that $r\mathbf{x}_1 + s\mathbf{x}_2$ is a solution.

37. Show that if $A\mathbf{x} = \mathbf{b}$ has more than one solution, then it has infinitely many solutions. (*Hint*: If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions, consider $\mathbf{x}_3 = r\mathbf{x}_1 + s\mathbf{x}_2$, where $r + s = 1$.)

38. Show that if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions to the linear system $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

39. Let

$$A = \begin{bmatrix} 6 & -1 & 1 \\ 0 & 13 & -16 \\ 0 & 8 & -11 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} 10.5 \\ 21.0 \\ 10.5 \end{bmatrix}.$$

(a) Determine a scalar r such that $A\mathbf{x} = r\mathbf{x}$.

(b) Is it true that $A^T\mathbf{x} = r\mathbf{x}$ for the value r determined in part (a)?

40. Repeat Exercise 39 with

$$A = \begin{bmatrix} -3.35 & -3.00 & 3.60 \\ 1.20 & 2.05 & -6.20 \\ -3.60 & -2.40 & 3.85 \end{bmatrix}$$

and $\mathbf{x} = \begin{bmatrix} 12.5 \\ -12.5 \\ 6.25 \end{bmatrix}$.

41. Let $A = \begin{bmatrix} 0.1 & 0.01 \\ 0.001 & 0.0001 \end{bmatrix}$. In your software, set the display format to show as many decimal places as possible, then compute

$$B = 10 * A,$$

$$C = \underbrace{A + A + A + A + A + A + A + A + A + A}_{10 \text{ summands}},$$

and

$$D = B - C.$$

If D is not O, then you have verified that scalar multiplication by a positive integer and successive addition are not the same in your computing environment. (It is not unusual that $D \neq O$, since many computing environments use only a "model" of exact arithmetic, called floating-point arithmetic.)

42. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. In your software, set the display to show as many decimal places as possible. Experiment to find a positive integer k such that $A + 10^{-k} * A$ is equal to A. If you find such an integer k, you have verified that there is more than one matrix in your computational environment that plays the role of O.