#2.2.6 (a) (b)

$$\begin{aligned} x + y + 2z + 3w &= 13 \\ x - 2y + z + w &= 8 \\ 3x + y + z - w &= 1 \end{aligned}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 1 & 2 & 3 & 13 \\ 1 & -2 & 1 & 1 & 8 \\ 3 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{\text{rref}} \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 8 \end{array}\right] \\ \begin{array}{ccccc} d & d & d & f & \end{array} \end{array}$$

$$\rightsquigarrow \quad \begin{aligned} x - w &= -2 \\ y &= -1 \\ z + 2w &= 8 \end{aligned}$$

$$\Longleftrightarrow \quad \begin{aligned} x &= w - 2 \\ y &= -1 \\ z &= -2w + 8 \end{aligned}$$

w free.

#2.2.20 $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -1 & 3 \\ 2 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Find x, y, z s.t. $f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 5 \\ -1 \end{bmatrix}$

Augment

$$\left[\begin{array}{ccc|c} 4 & 1 & 3 & 4 \\ 2 & -1 & 3 & 5 \\ 2 & 2 & 0 & -1 \end{array}\right] \xrightarrow{\text{rref}} \begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3/2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} d & d & f \end{array} \end{array}$$

$$\Longrightarrow \quad \begin{aligned} x &= -z + 3/2 \\ y &= -z - 2 \\ z &\text{ is free} \end{aligned}$$

pick $z = 0$

$$\Longrightarrow \quad \begin{aligned} x &= 3/2 \\ y &= -2 \end{aligned}$$

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/2 \\ -2 \\ 0 \end{bmatrix}$ is such a vector.

2.2.42

$$\left[\begin{array}{cc|c} 1-i & 2+2i & i \\ 1+i & -2+2i & -2 \end{array}\right] \quad (1-i)R_2 - (1+i)R_1$$

a_{22}: $-(2+2i)(1+i) + (-2+2i)(1-i)$

$= -(\cancel{2} + 2i + 2i - \cancel{2}) + (\cancel{-2} + 2i + 2i + \cancel{2})$

$= 0$

a_{21}: $-(1-i)(1+i) + (1+i)(1-i) = -|1+i|^2 + |1+i|^2 = 0$

a_{23}: $-(i)(1+i) + (-2)(1-i)$

$= -(i-1) + (-2+2i)$

$= -1+i$

$$\longrightarrow \left[\begin{array}{cc|c} 1-i & 2+2i & i \\ 0 & 0 & -1+i \end{array}\right]$$

system has no solution.

#2.3.10 (b)

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -3/2 & 5/2 & -1/2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ -3/2 & 5/2 & -1/2 \end{bmatrix}$$

#2.3.14 $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$

$$U_{31}(-1)A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -2 & 0 \end{bmatrix}$$

$$P_{23}U_{31}(-1)A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$U_{12}(1)P_{23}U_{31}(-1)A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

$$U_{32}(1/2)U_{12}(1)P_{23}U_{31}(-1)A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$D_2(-1/2)D_3(1/2)U_{32}(1/2)U_{12}(1)P_{23}U_{31}(-1)A$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U_{13}(-3)D_2(-1/2)D_3(1/2)U_{32}(1/2)U_{12}(1)P_{23}U_{31}(-1)A$$

$$= I_3$$

this proves A is invertible.

$\rightarrow$ cont.

$$\Rightarrow A = \left(U_{13}(-3) D_2(-1/2) D_3(1/2) U_{32}(1/2) U_{12}(1) P_{23} U_{31}(-1) \right)^{-1} I_3$$

$$\Rightarrow A = U_{31}(-1)^{-1} P_{23}^{-1} U_{12}(1)^{-1} U_{32}(1/2)^{-1} D_3(1/2)^{-1} D_2(-1/2)^{-1} U_{13}(-3)^{-1}$$

$$A = U_{31}(1) P_{23} U_{12}(-1) U_{32}(-1/2) D_3(2) D_2(-2) U_{13}(3)$$