4.2.12

$$V = \{ x \in \mathbb{R} \mid x > 0 \}$$

$$x \oplus y = xy$$

$$c \odot x = x^c$$

(A1) Let $x, y \in V$. Then

$$x \oplus y = xy = yx = y \oplus x.$$

(A2) Let $x, y, z \in V$. Then

$$\begin{aligned} x \oplus (y \oplus z) &= x \oplus (yz) \\ &= x(yz) \\ &= (xy)z \\ &= (xy) \oplus z \\ &= (x \oplus y) \oplus z \end{aligned}$$

(A3) $\vec{0} = 1$. Let $x \in V$. Then

$$x \oplus \vec{0} = x \oplus 1 = x \cdot 1 = x$$

(A4) Let $x \in V$. Then $-x = \frac{1}{x}$.

$$x \oplus -x = x \oplus \frac{1}{x} = x \cdot \frac{1}{x} = 1 = \vec{0}$$

(A5) Let $c \in \mathbb{R}$ and $x, y \in V$. Then

$$\begin{aligned} c \odot (x \oplus y) &= c \odot xy \\ &= (xy)^c \\ &= x^c y^c \\ &= x^c \oplus y^c \\ &= (c \odot x) \oplus (c \odot y) \end{aligned}$$

(A6) Let $c, d \in \mathbb{R}$ and $x \in V$. Then

$$\begin{aligned} (c+d) \odot x &= x^{c+d} \\ &= x^c \cdot x^d \\ &= x^c \oplus x^d = (c \odot x) \oplus (d \odot x) \end{aligned}$$

(A7) Let $c, d \in \mathbb{R}$ and $x \in Y$. Then

$$c \odot (d \odot x) = c \odot x^d$$

$$= (x^d)^c$$

$$= x^{cd}$$

$$= (cd) \odot x$$

(A8) Let $x \in Y$. Then

$$1 \odot x = x^1 = x.$$

#4.3.18

(a) $W = \{A \in M_n(\mathbb{R}) : A \text{ is singular}\}$

$= \{A \in M_n(\mathbb{R}) : \det(A) = 0\}$

Not closed under addition. Let A be the matrix obtained from I_n by replacing the 1,1-entry with 0., and let B be the matrix w/ a 1 in the 1,1-entry and zeros everywhere else. Then A and B are in W since both have determinant 0.

But $A + B = I_n$ has determinant $1 \neq 0$

So $A + B \notin W$.

$\therefore$ not a subspace

(b) $W = \{ A \in M_n(\mathbb{R}) \mid A \text{ is upper triangular} \}$

Closure under $+$: Let $A, B \in W$. Then

$$A = \begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \diagdown & \star \\ 0 & \diagdown \end{bmatrix}$$

$$\Longrightarrow \quad A + B = \begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix} + \begin{bmatrix} \diagdown & \star \\ 0 & \diagdown \end{bmatrix}$$

$$= \begin{bmatrix} \diagdown & * + \star \\ 0 & \diagdown \end{bmatrix} \text{ is upper triangular.}$$

$$\therefore \quad A + B \in W$$

Closure under $\cdot$: Let $c \in \mathbb{R}$ and $A \in W$.
Then A has the form

$$A = \begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix}$$

Then

$$cA = c\begin{bmatrix} \diagdown & * \\ 0 & \diagdown \end{bmatrix} = \begin{bmatrix} \diagdown & c* \\ 0 & \diagdown \end{bmatrix}$$

is upper triangular

$\therefore\ cA \in W$

Since W is closed under $+$ and $\cdot$ it is a subspace

(c) $W = \{ A \in M_n(\mathbb{R}) \mid A^T = -A \}$

Closure under $+$: Let $A, B \in W$. Then

$A^T = -A$ and $B^T = -B$.

So that

$$(A+B)^T = A^T + B^T = -A - B$$
$$= -(A+B)$$

$\therefore A+B \in W$.

Closure under $\cdot$: Let $c \in \mathbb{R}$ and $A \in W$.

Then $A^T = -A$. So that

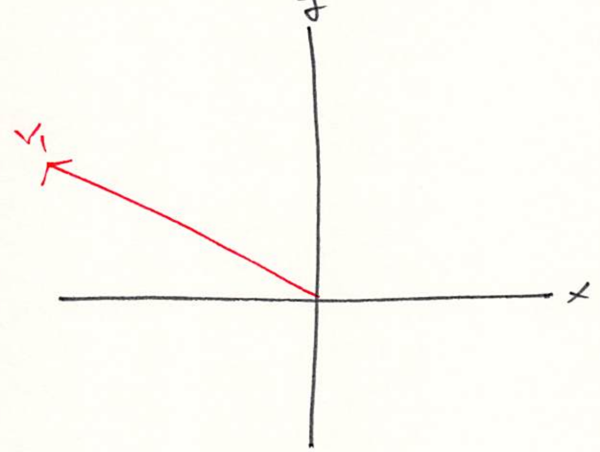
$$(cA)^T = cA^T = c(-A) = -(cA)$$

$\therefore \alpha A \in W$

Hence, W is a subspace.

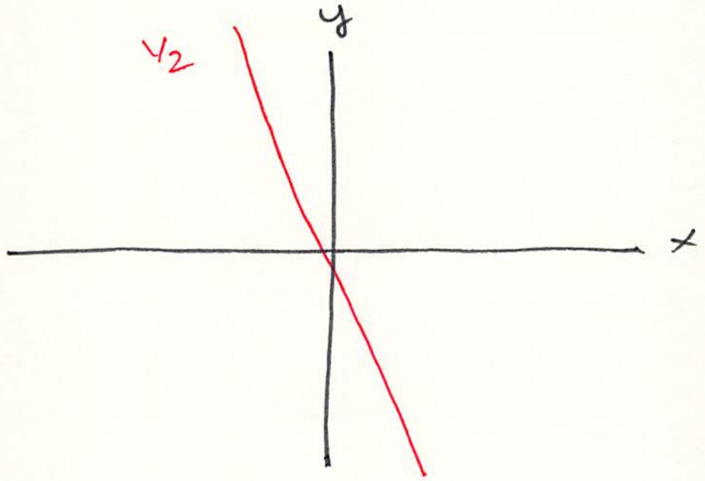
#4.3.30

(a)



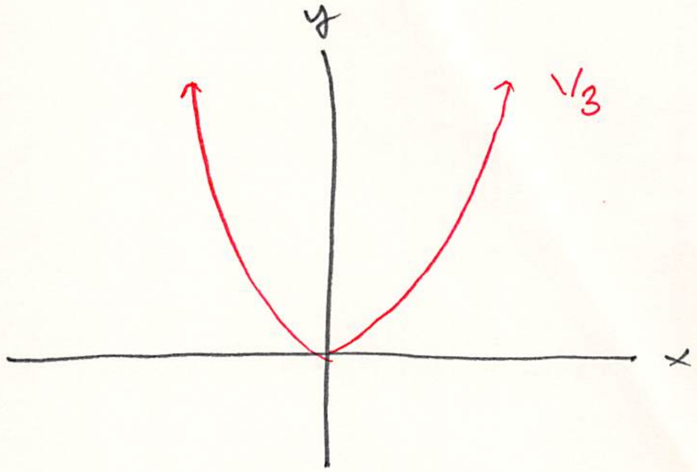
V_1 is closed under addition, but not closed under scalar mult. If $v \in V_1$, then $-v$ is not.

(b)



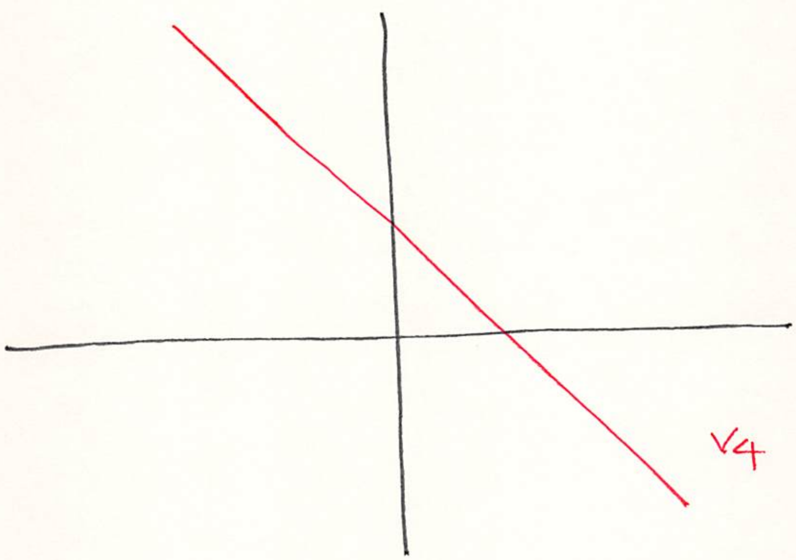
V_2 is a subspace it is both closed under addition and scaler mult.

(c)



V_3 is not a subspace not closed under addition and not closed under scalar mult.

(d)



V_4 is not a subspace, not closed under addition and not closed under scalar mult.