4.4.6

(b)

$$c_1 \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \overset{\substack{\text{arbitrary} \\ \downarrow}}{\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}}$$

$\Longleftrightarrow$ the linear system

$$\left[\begin{array}{ccc|c} 3 & 1 & 0 & a \\ 2 & 2 & 0 & b \\ 1 & -1 & 0 & c \\ 0 & 0 & 1 & d \end{array}\right]$$

has a solution

$$\xrightarrow{\text{MATLAB}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & b/4 + c/2 \\ 0 & 1 & 0 & b/4 - c/2 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & a - b - c \end{array}\right]$$

The system only has solution if $a - b - c = 0$, but a, b, c were arbitrary, so the vectors do not span $\mathbb{R}^4$.

(d) there are scalars c_1, c_2, c_3, c_4 s.t.

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} + c_4 \begin{bmatrix} 2 \\ 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

↑ arbitrary in $\mathbb{R}^4$

$\iff$ the linear system

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 2 & a \\ 1 & 2 & 0 & 1 & b \\ 0 & -1 & 1 & 2 & c \\ 0 & 1 & -1 & -1 & d \end{array}\right]$$

has a solution.

$$\xrightarrow{\text{MATLAB}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2a - b - 3c - 3d \\ 0 & 1 & 0 & 0 & b - a + c + d \\ 0 & 0 & 1 & 0 & b - a - d \\ 0 & 0 & 0 & 1 & c + d \end{array}\right]$$

the system has a unique solution for any a, b, c, d, so the vectors span $\mathbb{R}^4$.

4.~~5~~.13. Let

$$W = \{ A \in M_2(\mathbb{R}) : \operatorname{tr}(A) = 0 \}$$

and let

$$v_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Set $S = \{v_1, v_2, v_3\}$. Show $\operatorname{span}(S) = W$.

Pf. Let $A \in W$. Then if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

we have

$$\operatorname{tr}(A) = a + d = 0 \implies d = -a.$$

Hence, A has the form $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$.

So

$$A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix} = b v_1 + c v_2 + a v_3$$

So S spans W. □

4.5.15

(a)

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a unique solution

$\Longleftrightarrow$ the linear system

$$\left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{matrix}}_{A} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right]$$ has a unique solution

$\det(A) = -3 \neq 0$

Matlab $\nearrow$

$\Longrightarrow$ A is invertible

$\Longrightarrow$ system has a unique solution

$\Longrightarrow$ vectors are L.I.

(b) Solutions to the equation

$$c_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

are solutions to the linear system

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 & 0 \\ -1 & 1 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{\text{MATLAB}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

c_4 is free

$c_1 = -2c_4$

$c_2 = -c_4$

$c_3 = c_4$

Pick $c_4 = 1 \Longrightarrow c_1 = -2$

$c_2 = -1$

$c_3 = 1$

$\longrightarrow$

Then

$$(-2)\begin{bmatrix}1\\1\\-1\end{bmatrix}+(-1)\begin{bmatrix}0\\1\\1\end{bmatrix}+(1)\begin{bmatrix}1\\1\\1\end{bmatrix}+(1)\begin{bmatrix}1\\2\\-2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}0\\1\\1\end{bmatrix}=(-2)\begin{bmatrix}1\\1\\-1\end{bmatrix}+(1)\begin{bmatrix}1\\1\\1\end{bmatrix}+(1)\begin{bmatrix}1\\2\\-2\end{bmatrix}$$