Introduction

Proof writing is a fundamental skill in mathematics, students typically begin to develop this skill in a sophomore level course in linear algebra. Proof writing involves presenting a logical argument to demonstrate the validity of a mathematical statement. The following is a very short guide to help you write clear and rigorous proofs in our course.

Golden Rule of Proof Writing

> *You are writing for someone else, make it as easy as possible for a reader to understand your work.*

Structure of a Proof

The structure of a proof is very similar to an essay you wrote in high school, there is thesis, introduction, body and conclusion.

1. **Thesis**: Clearly state what is being proved. Let the reader know what you are about to show.

2. **Introduction**: Set expectations, list the information given by the problem, declare any necessary variables, maybe restate what you will attempt to show in your new variables, etc.

3. **Body**: Present the logical sequence of statements that lead to your conclusion.

4. **Conclusion**: Explain to the reader how you've proven the result. It is traditional to place a QED or $\square$ at the end your proof, this way proofs are self-contained.

Direct Proofs

There a many proof techniques in mathematics: direct proofs, proof by contradiction, proofs by induction just to name a few. In our class you will be able to answer almost all proof questionis with direct proofs. We use direct proofs when the structure of the statement we are trying to prove has the form "If P, then Q" or "P implies Q", in symbols, $P \Rightarrow Q$. The strategy for proving a statement of this type is simple:

(i) Assume the hypothesis P is true.

(ii) Demonstrate that the conclusion Q follows using any mathematics available to you.

Style Tips

1. **Clarity**: Write out mathematical steps clearly and justify complicated steps. Your math calculations should be neatly presented and simple to follow.
2. **Notation** Use standard mathematical notation, and be consistent with it. Avoid unnecessary notation, do not introduce variables just for the sake of it.
3. **Grammar and Punctuation**: Use complete English sentences and punctuation. Read your proof out loud, it should make sense.
4. **Logical Flow**: Your argument should follow a logical sequence. Do not begin with your conclusion. Often you have to do a lot of scratch work to unlock a proof. Do not turn in your scratch work, clean it up, organize it logically, and tell a story.

Examples

Next is an example of proof for the same statement "*if n is even, then n^2 is even*" that avoids all the guidelines above.

Example 1. *Prove that if n is even, then n^2 is even.*

Proof. $n^2 = (2k)^2 = 4k^2$. □

That is technically a proof of the statement. It's OK if you are confused, that's normal, the proof is not a very good one. Read the proof out loud, how would anyone know this is discussing the evenness of n^2? Below is a better example of direct proof of the statement that follows the guidelines discussed above.

Example 2. *Prove that if n is even, then n^2 is even.*

Proof. Recall that an integer is even if it is divisible by 2. Suppose that n is even. So we can write $n = 2k$ for some integer k. Then

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 2(2k^2) \\ &= 2k', \text{ where } k' = 2k^2. \end{aligned}$$

Since k is an integer, then so is k'. Hence, n^2 is divisible by 2 as $n^2 = 2k'$. Therefore, n^2 is even. □

The proof follows the strategy of a direct proof: it begins by assumming the hypothesis "n is even", and deduces the conclusion "n^2 is even" through a logical sequence statements.

Observe the structure of the proof. The first paragraph is the introduction. It reviews the definition of an even integer for the reader, it declares n to be even, and introduces the auxilliary variable k. The sequence of equalities in the middle is the body. Each equality performs a simple algebra step that is easy to follow. The last paragraph is the conclusion. A secondary variable k' is introduced that is meant help explain why n^2 is also divisilble by 2.

Notice how the proof agrees with the style tips. The math steps are organized, only necessary notation is introduced, there are only complete sentences and no fragments, and the proof begins with the hypothesis and ends with the conclusion. Read the proof out loud, it reads like an explanation. A person with a high school level understanding of algebra should be able to read and understand this proof.

Of course there is more that one way to write a proof. The following is a proof for the same statement. The difference is the proof is less verbose and assumes the reader has more mathematical training. Note the proof still meets all the guidelines discussed above.

Example 3. *Prove that if n is even, then n^2 is even.*

Proof. Suppose $n = 2k$ for some k. Then $n^2 = 2(2k^2)$, hence, n^2 is even. □

For our course you should strive to write proofs that someone else in our class can pick up and read easily. Wrtiting proofs is an art. This guide is a crash course on the presentation of one type of proof method, it doesn't address how to find proofs.