Math 311 Exam 2 Review, Fall 2024

Name:

Question	Points	Score
1	0	
2	0	
3	0	
4	0	
5	0	
6	0	
7	0	
8	10	
Total:	10	

- You have 75 minutes to complete this exam.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work, unless otherwise indicated. You will get almost no credit for solutions that are not fully justified.
- No electronic devices are authorized.
- You may use a 3x5 notecard with notes, no other resources are authorized.
- The back side of each page can be used as scratch work or for longer solutions. Please indicate clearly if your solution extends to the back side of a page.

1. It's perfectly fine to use computational software for the tedious algebra in this review! Just remember that you have to do these by hand on the actual exam.

2. For the following pairs A and $\vec{b}$ answer the following.
   (a) Find bases for the null space, column space, and row space of A.
   (b) Find the general solution to the nonhomogeneous system $A\vec{x} = \vec{b}$. Write your answer in the form $\vec{x} = \vec{x}_h + \vec{x}_p$.
   (c) Find rank(A) and nullity(A).

(i) $A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 2 & 3 & 5 & 7 \\ -1 & -2 & -3 & -4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 11 \\ 13 \\ -8 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ 2 & 5 & 8 & 11 & 14 \\ -1 & -2 & -3 & -4 & -5 \\ 1 & 5 & 9 & 13 & 17 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 10 \\ 16 \\ -7 \\ 13 \end{bmatrix}$

3. Find a basis for the span of the following lists of vectors in their corresponding vector spaces.

(i)
$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \\ 11 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ -4 \end{bmatrix} \right\}$$

(ii)
$$\left\{ \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 5 & 7 & 11 \end{bmatrix}, \begin{bmatrix} 1 & 4 & 9 & 16 & 25 \end{bmatrix} \right\}$$

(iii)
$$\{1 + t + t^2,\ 1 - t + 2t^2,\ 3 + 4t + 5t^2,\ 2 - 3t + 4t^2\}$$

Hint: transfer the vectors to $\mathbb{R}^4$ with the coordinate map w.r.t. the standard basis, find a basis for the span of the coordinate vectors, then use the inverse coordinate map to obtain a basis in P_2.

4. Below your are given lists of vectors S and T in a vector space V and a vector $v \in V$. Answer the following.
   (a) Show that S and T are bases of V.
   (b) Find the transition matrices $P_{S \leftarrow T}$ and $P_{T \leftarrow S}$.
   (c) Find the coordinate vector $[v]_S$.
   (d) Use a transition matrix to get $[v]_T$.

(i)

$$
\begin{aligned}
V &= \mathbb{R}^3, \\
S &= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}, \\
T &= \left\{ \begin{bmatrix} -4 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 17 \\ 16 \\ 7 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 2 \end{bmatrix} \right\}, \\
v &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&V = P_2 \\
&S = \left\{2t^2 + t,\ 3t + 1, -t^2 + 2t + 4\right\}, \\
&T = \left\{3t^2 + 2t - 3,\ \ -4t^2 - 5t - 1,\ \ -6t^2 + t + 14\right\}, \\
&v = t - 1
\end{aligned}
$$

(iii)

V is an abstract vector space

$S = \{v_1, v_2, v_3\}$ (assume this is a basis)

$T = \{-v_2 + 2v_3,\ \ -v_1 - v_2 + 2v_3,\ \ -v_1 - v_2 + v_3\},$

$v = v_1 - 2v_3$

5. Determine if the following sets W are subspaces.

(i) $W = \{p \in P_2 : p(-t) = -p(t)\}$

(ii) $W = \{p \in P_2 : \int_{-1}^{1} p(t)dt = 0\}$

(iii) $W = \{p \in P_2 : p''(t) + p'(t) = 1\}$

(iv) $W = \{\{a_n\}_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : a_n \text{ is odd for all } n\}$

(v) $W = \{\{a_n\}_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \lim_{n\to\infty} a_n = 0\}$

(vi) $W = \{\{a_n\}_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^{\infty} a_n \text{ converges}\}$

(vii) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + 2y + 3z = 0 \right\}$

(viii) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x - 2y = 1 \right\}$

(ix) $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : xyz = 0 \right\}$

6. Below you are given vectors u and v in an inner product space V. Determine the following
   (a) Find $\|u\|$, $\|v\|$, and $\text{dist}(u, v)$.
   (b) Find $\langle u, v\rangle$. Are the vectors orthogonal?
   (c) What is the angle between u and v?

   (i) The vectors $u = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ in the inner product space $\mathbb{R}^3$ with the standard inner product.
   (ii) Let $u = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$ in the inner product space $M_{2\times 3}(\mathbb{R})$ with inner product $\langle A, B\rangle = \text{tr}(A^T B)$
   (iii) The vectors $u = \sin t$ and $v = \cos t$ in the inner product space $C[0, 2\pi]$ with inner product $\langle f, g\rangle = \int_0^{2\pi} f(t)g(t)dt$.

7. Below you are given a family of matrices and vectors parametrized by t.

$$A(t) = \begin{bmatrix} e^t & \sin t & \cos t \\ e^t & \cos t & -\sin t \\ e^t & -\sin t & -\cos t \end{bmatrix}, \quad \vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, \quad \vec{b}(t) = \begin{bmatrix} 0 \\ 0 \\ t \end{bmatrix}.$$

   For example, at $t = 0$ we have

$$A(0) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} x(0) \\ y(0) \\ z(0) \end{bmatrix}, \quad \vec{b}(t) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

   You can think of $A(t)$ as path of matrices in the space of 3×3 matrices, $\vec{b}(t)$ is the path in $\mathbb{R}^3$ that travels up the z-axis, and $\vec{x}(t)$ is an unknown path in $\mathbb{R}^3$.
   (a) Show that $\det(A(t)) = -e^t$. Conclude that $A(t)$ is invertible for all t.
   (b) Since $A(t)$ is invertible apply Cramer's Rule to find the unknown path $\vec{x}(t)$ that satisfies the linear system $A(t)\vec{x}(t) = \vec{b}(t)$.

8. (10 points) True/False questions. No justification necessary.
   (a) True False There is a real vector space V such that $2v = \vec{0}$ for all $v \in V$.
   (b) True False Every plane in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

(c) True False There is a redundant vector in the list

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

(d) True False The polynomials $t^2 + 3t + 1$ and $2t^2 - 1$ span P_2.

(e) True False If a vector space has dimension at least 2, then the vector space contains infinitely many different subspaces of dimension 1.

(f) True False $P_6 \cong M_{2\times 3}(\mathbb{R})$.

(g) True False The row space of a matrix A is isomorphic to the column space of the matrix A.

(h) True False A 5×3 matrix can have nullity 2 and rank 3.

(i) True False If S is a basis of V, then the coordinate map $\phi_S : V \to \mathbb{R}^n$ is an isomorphism.

(j) True False If S, T and U are bases for a vector space V, then the transition matrices satisfy $P_{S\leftarrow T}P_{T\leftarrow U} = P_{S\leftarrow U}$.