Math 311 Final, Fall 2024

Name:

Question	Points	Score
1	0	
2	0	
3	0	
4	0	
5	0	
6	0	
7	0	
8	0	
9	0	
Total:	0	

- You have 120 minutes to complete this exam.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work, unless otherwise indicated. You will get almost no credit for solutions that are not fully justified.
- No electronic devices are authorized.
- You may use a 5x8 notecard with notes, no other resources are authorized.
- The back side of each page can be used as scratch work or for longer solutions. Please indicate clearly if your solution extends to the back side of a page.

1. Solve linear system. Write you answer in vector form $\vec{x}_h + \vec{x}_p$.

$$\begin{aligned} x_1 - 3x_2 + x_3 - x_4 - x_5 &= 1 \\ 2x_1 + x_2 - x_3 + 2x_4 + x_5 &= 2 \\ -x_1 + 3x_2 - x_3 - 2x_4 - x_5 &= 3 \\ 2x_1 + x_2 - x_3 - x_4 - x_5 &= 6 \end{aligned}$$

2. Determinants.

$$A = \begin{bmatrix} -3 & 0 & 4 \\ 2 & 1 & -3 \\ 4 & 0 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & -1 & 3 \\ 4 & 1 & -2 \\ -3 & 2 & 1 \end{bmatrix}$$

Find $\det(A)$, $\det(B)$, $\det(-A^T)$, $\det((B^{-1})^2)$

3. Determine if the following are subspaces.
   (a) $W = \{f \in F[a,b] : \ f(a) = 0\}$
   (b) $W = \{f \in F[a,b] : \ f(a) = 1\}$
   (c) $W = \{A \in M_n(\mathbb{R}) : \ A \text{ is diagonal}\}$
   (d) $W = \{A \in M_n(\mathbb{R}) : \ \det(A) = 0\}$

4. Answer the following
   (i) Is S linearly dependent or independent?
   (ii) Is the vector v in the span of S.
   (iii) Does S form a basis for V.
   (a) $V = \mathbb{R}^3$, $S = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$, $v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$
   (b) $V = \mathbb{R}^3$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}$, $v = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$
   (c) $V = P_2$, $S = \{x^2 - 3x, x + 7\}$, $v = 7x^2 - 25x + 28$.
   (d) $V = P_2$, $S = \{x^2 + x + 1, x^2 - x + 1, x^2 - 1\}$, $v = 3x^2$

5. Use the Gram-Schmidt process to transform S into an orthonormal basis.
   (a) $V = \mathbb{R}^3$ with the usual dot product, $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right\}$.
   (b) $V = P_2$ with inner product $\langle p, q \rangle = \int_{-1}^{1} p(t)q(t)dt$, and $S = \{1, t, t^2\}$ (in this order!). The resulting polynomials when we apply this process to $1, t, \ldots, t^n$ are called normalized Legendre polynomials.

6. Find the orthogonal complement $W^{\perp}$ of $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 3 \end{bmatrix} \right\}$

7. Determine if the following maps are linear transformations.
   (a) $L : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ where $L(A) = A^2$
   (b) $L : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ where $L(A) = \dfrac{A + A^T}{2}$
   (c) $L : D^2(a, b) \to F(a, b)$ where $L(f(t)) = f''(t) + tf'(t) + t^2 f(t)$

8. Let $L : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be the transpose map $L(A) = A^T$. Let

$$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$T = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

   (a) Find $P_{S \leftarrow T}$ and $P_{T \leftarrow S}$.
   (b) Find $[L]_S^S$.
   (c) Use parts (a) and (b) to find $[L]_T^T$.
   (d) Find the eigenvalues of L and bases for its eigenspaces.

9. For the following matrices

   (i) Find the eigenvalues of A.
   (ii) Compute bases for the eigenspaces of A.
   (iii) Is the matrix A diagonalizable, if so find a diagonal matrix D and an invertible matrix P such that $D = P^{-1}AP$.

   (a) $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 0 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$