Math 242 Final Fall 2016

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Question	Points	Score
1	7	
2	10	
3	12	
4	40	
5	5	
6	32	
7	10	
8	6	
9	6	
10	6	
11	16	
Total:	150	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. (7 points) Say $g(x) = e^{3x-2}$. Find, and simplify, a formula for $g^{-1}(x)$.

$$x = e^{3y-2}$$

$$\ln x = 3y - 2$$

$$2 + \ln x = 3y$$

$$\frac{2+\ln x}{3} = y$$

$$g^{-1}(x) = \frac{2+\ln x}{3}$$

2. Compute the derivatives of the following functions. You do not have to simplify your answers.

(a) (5 points) $f(x) = 2^{2x} + \ln(7)$.

$$f'(x) = 2^{2x} \ln 2 \cdot 2 + 0$$

(b) (5 points) $f(x) = (\sin x)^x$.

$$y = (\sin x)^x$$

$$\ln y = x \ln(\sin x)$$

$$\frac{1}{y} y' = 1 \ln(\sin x) + x \frac{\cos x}{\sin x}$$

$$y' = y\left(\ln(\sin x) + \frac{x \cos x}{\sin x}\right)$$

$$f'(x) = (\sin x)^x \left(\ln(\sin x) + \frac{x \cos x}{\sin x}\right)$$

3. Compute the following limits. You must justify your solution using algebraic manipulations and / or l'Hôpital's rule for full credit.

(a) (6 points) $\lim_{n\to\infty} \frac{\ln n}{e^n}$.

Limit type: $\frac{\infty}{\infty}$

May use L'H Rule

$\stackrel{H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{e^n}$

Limit type $\frac{0}{\infty} \to 0$

$= 0$

(b) (6 points) $\lim_{x\to 0^+} (3/x)^x$.

Limit type ∞^0

Indeterminate

Set $L = \lim_{x\to 0^+} \left(\frac{3}{x}\right)^x$

$\ln L = \lim_{x\to 0^+} x \ln\left(\frac{3}{x}\right)$ Type $0 \cdot \infty$.

$= \lim_{x\to 0^+} \frac{\ln\left(\frac{3}{x}\right)}{1/x}$ Type $\frac{\infty}{\infty}$ Use L'H Rule

$\stackrel{H}{=} \lim_{x\to 0^+} \frac{1/\frac{3}{x} \cdot \frac{-3}{x^2}}{-\frac{1}{x^2}}$

$= \lim_{x\to 0^+} \frac{x}{3} \cdot \frac{-3}{x^2} \cdot \frac{-x^2}{1} = \lim_{x\to 0^+} x = 0$

Then $L = e^0 = \boxed{1}$

4. Compute the following integrals, or say if they diverge.

(a) (10 points) $\int \sin^5(x)dx$.

ODD power of sin.

Let $u = \cos x$

$du = -\sin x\, dx$

$-du = \sin x\, dx$

$1-u^2 = 1-\cos^2 x$

$$= \int \sin^4(x) \quad \sin(x)\, dx$$

$$= \int (\sin^2 x)^2 \quad \sin x\, dx$$

$$= \int (1-\cos^2 x)^2 \sin x\, dx$$

$$= \int (1-u^2)^2 (-du)$$

$$= -\int (1-2u^2+u^4)\, du$$

$$= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C$$

$$= -\left(\cos x - \frac{2}{3}\cos^3 x + \frac{1}{5}\cos^5 x\right) + C$$

(b) (10 points) $\int x \ln(\sqrt{x})dx$.

$u = \ln(\sqrt{x})$ $\quad dv = x\,dx$

$du = \frac{1}{\sqrt{x}} \frac{1}{2} x^{-1/2} dx$ $\quad v = \frac{1}{2}x^2$

$= \frac{1}{2x} dx$

$= \frac{1}{2}x^2 \ln(\sqrt{x}) - \int \frac{1}{2}x^2 \cdot \frac{1}{2x} dx$

$- \int \frac{1}{4} x\,dx$

$- \frac{1}{8}x^2 + C$

$= \frac{1}{2}x^2 \ln(\sqrt{x}) - \frac{1}{8}x^2 + C$

other approaches possible such as noticing that $\ln(\sqrt{x}) = \frac{1}{2}\ln x$

(c) (10 points) $\int_3^\infty \frac{1}{2x^2-x-1}dx.$

$$\frac{1}{2x^2-x-1} = \frac{1}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}$$

$$1 = A(x-1) + B(2x+1)$$

Set $x=1$: $1 = 0 + B(3) \Rightarrow B = 1/3$

Set $x=-1/2$: $1 = A(-3/2)+0 \Rightarrow A = -2/3$

$$\int \frac{1}{2x^2-x-1}dx = \int\left(\frac{-2/3}{2x+1} + \frac{1/3}{x-1}\right)dx$$

$$= -\frac{2}{3}\frac{\ln|2x+1|}{2} + \frac{1}{3}\ln|x-1|.$$

$$= \lim_{b\to\infty}\int_3^b \frac{1}{2x^2-x-1}dx$$

$$= \lim_{b\to\infty}\left[-\frac{1}{3}\ln|2x+1| + \frac{1}{3}\ln|x-1|\right]_3^b$$

$$= \lim_{b\to\infty}\left[-\frac{1}{3}\ln(2b+1) + \frac{1}{3}\ln(b-1) - \left(-\frac{1}{3}\ln 7 + \frac{1}{3}\ln 2\right)\right]$$

$$= \lim_{b\to\infty}\left[-\frac{1}{3}\ln(2b+1) + \frac{1}{3}\ln(b-1)\right] + \frac{1}{3}\ln 7 - \frac{1}{3}\ln 2$$

$$= \lim_{b\to\infty}\frac{1}{3}\left[-\ln(2b+1) + \ln(b-1)\right] + \frac{1}{3}\ln 7 - \frac{1}{3}\ln 2$$

$$= \lim_{b\to\infty}\frac{1}{3}\ln\left(\frac{b-1}{2b+1}\right) + \frac{1}{3}\ln 7 - \frac{1}{3}\ln 2$$

$$= \frac{1}{3}\ln\left(\frac{1}{2}\right) + \frac{1}{3}\ln 7 - \frac{1}{3}\ln 2$$

CAN FURTHER SIMPLIFY $= \frac{1}{3}\ln\left(\frac{7}{4}\right).$

(d) (10 points) $\int_0^{\frac{1}{4}} \sqrt{1-4t^2}\,dt$.

$2t = \sin\theta$

so that $\sqrt{1-(2t)^2} = \cos\theta$

$t = \frac{1}{2}\sin\theta$

$dt = \frac{1}{2}\cos\theta\, d\theta$

$\sin^{-1}(2t) = \theta$

$$\int\sqrt{1-4t^2}\,dt$$

$$= \int \cos\theta \cdot \frac{1}{2}\cos\theta\, d\theta$$

$$= \frac{1}{2}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{4}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{1}{4}\left(\sin^{-1}(2t) + 2t\sqrt{1-4t^2}\right) + C$$

$$\int_0^{1/4}\sqrt{1-4t^2}\,dt$$

$$= \frac{1}{4}\left(\sin^{-1}(2t) + 2t\sqrt{1-4t^2}\right)\Big|_0^{1/4}$$

$$= \frac{1}{4}\left(\sin^{-1}\left(\frac{1}{2}\right) + 2\cdot\frac{1}{4}\sqrt{1-\frac{1}{4}}\right) - \frac{1}{4}\left(\sin(0) + 0\right)$$

$$= \frac{1}{4}\left(\frac{\pi}{6} + \frac{1}{2}\sqrt{\frac{3}{4}}\right).$$

5. (5 points) Compute the sum of the convergent series $\sum_{n=0}^{\infty} \frac{3^{n-1}}{4^n}$. Simplify your answer.

Geometric series.

1^{ST} term $= \frac{3^{-1}}{4^0} = \frac{1}{3}$

ratio $= \frac{3}{4}$.

$$\text{sum} = \frac{\frac{1}{3}}{1-\frac{3}{4}} = \frac{\frac{1}{3}}{\frac{1}{4}} = \frac{4}{3}.$$

6. For each of the following series, say whether they converge or diverge. For full credit, you must justify your solutions, and state clearly which test(s) you are using (if any).

(a) (8 points) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^{4/3}}$.

$$\frac{4}{3} - \frac{1}{2} = \frac{8}{6} - \frac{3}{6} = \frac{5}{6}$$

$$= \sum_{n=1}^{\infty} \frac{n^{1/2}}{n^{4/3}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{4/3-1/2}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/6}}$$

p-series

$p = \frac{5}{6} < 1.$

Diverges.

(b) (8 points) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2+n}$. Alternating series $u_n = \frac{\sqrt{n}}{2+n}$.

- Decreasing? $\frac{d}{dn} u_n = \frac{\frac{1}{2}n^{-1/2}(2+n) - \sqrt{n}(1)}{(2+n)^2} = \frac{n^{-1/2} + \frac{1}{2}\sqrt{n} - \sqrt{n}}{(2+n)^2}$

$$= \frac{n^{-1/2} - \frac{1}{2}\sqrt{n}}{(2+\sqrt{n})^2} \cdot \frac{\sqrt{n}}{\sqrt{n}} = \frac{1 - \frac{1}{2}n}{\sqrt{n}\,(2+\sqrt{n})^2}$$

So $\frac{d}{dn} u_n \leq 0$ for $n \geq 2$ and so u_n is decreasing.

- $\lim_{n\to\infty} u_n \underset{\text{Type}\frac{\infty}{\infty}}{\overset{L'H}{=}} \lim_{n\to\infty} \frac{\frac{1}{2}n^{-1/2}}{1} = 0$

By alternating series test, the series converges.

(c) (8 points) $\sum_{n=0}^{\infty} \frac{2^n}{(n+2)!}$. positive series

Ratio test. $\rho = \lim_{n \to \infty} \frac{2^{n+1}}{(n+3)!} \Big/ \frac{2^n}{(n+2)!}$

$= \lim_{n \to \infty} \frac{2^{n+1}}{2^n} \frac{(n+2)!}{(n+3)!} = \lim_{n \to \infty} \frac{2}{n+3} = 0 < 1$

So the series converges.

(d) (8 points) $\sum_{n=1}^{\infty} \frac{\cos(n!)}{1+n^2}$. ← unpredictable $\pm$. Try absolute convergence.

$\left| \frac{\cos(n!)}{1+n^2} \right| \le \frac{1}{1+n^2} \le \frac{1}{n^2}$.

$\sum \frac{1}{n^2}$ converges (p-series $p=2$).

So $\sum \left| \frac{\cos(n!)}{1+n^2} \right|$ converges.

So $\sum \frac{\cos(n!)}{1+n^2}$ converges absolutely.

7. (10 points) Find the values of x for which the power series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}2^n}(x-1)^n$$

(a) converges absolutely; (b) converges conditionally; (c) diverges. Justify each answer.

Absolute ratio test.

$$\rho = \lim_{n\to\infty} \left| \frac{(X-1)^{n+1}}{\sqrt{n+1}\,2^{n+1}} \Big/ \frac{(X-1)^n}{\sqrt{n}\,2^n} \right| = \lim_{n\to\infty} \left| \frac{(X-1)^{n+1}}{(X-1)^n} \frac{\sqrt{n}}{\sqrt{n+1}} \frac{2^n}{2^{n+1}} \right|$$

$$= \lim_{n\to\infty} \left| (X-1)\sqrt{\frac{n}{n+1}}\,\frac{1}{2} \right| = |x-1|\cdot 1 \cdot \tfrac{1}{2} = \tfrac{1}{2}|X-1|.$$

Solve $\rho < 1 \Leftrightarrow \frac{1}{2}|X-1| < 1 \Leftrightarrow |x-1| < 2 \Leftrightarrow 1-2 < X < 1+2$

Radius of convergence is $R=2$.

Converges absolutely (so far) for $-1 < X < 3$.

Endpoints $\boxed{X=-1}$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,2^n}(-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

Alternating series $u_n = \frac{1}{\sqrt{n}}$ • decreasing obviously

• $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

Converges.

$\boxed{X=3}$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,2^n}(2)^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges. (p-series $p=\frac{1}{2}<1$)

This also says power series converges conditionally at $X=-1$.

(a) converges absolutely for $-1 < X < 3$

(b) converges conditionally for $X=-1$

(c) diverges otherwise.

8. (6 points) Find the order two Taylor polynomial for $\sqrt{x}$, centered at $a = 1$.

$$f(x) = \sqrt{x} \qquad f(1) = 1$$
$$f'(x) = \tfrac{1}{2} x^{-1/2} \qquad f'(1) = \tfrac{1}{2}$$
$$f''(x) = -\tfrac{1}{4} x^{-3/2} \qquad f''(1) = -\tfrac{1}{4}$$

$$P_2(x) = \frac{1}{0!} + \frac{\frac{1}{2}}{1!}(x-1) - \frac{\frac{1}{4}(x-1)^2}{2!}$$
$$= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2$$

9. (6 points) The function $f(x) = \cos(3x)$ is sometimes approximated by its second-order Taylor polynomial, $P_2(x) = 1 - 9x^2/2$, for small values of x.

Use a technique from the course to give a bound to the error of this estimate on the interval $[0, 0.1]$.

$$f'(x) = -3\sin(3x) \qquad \boxed{\begin{aligned} a &= 0 \\ n &= 2 \end{aligned}}$$
$$f''(x) = -9\cos(3x)$$
$$f'''(x) = 27\sin(3x)$$

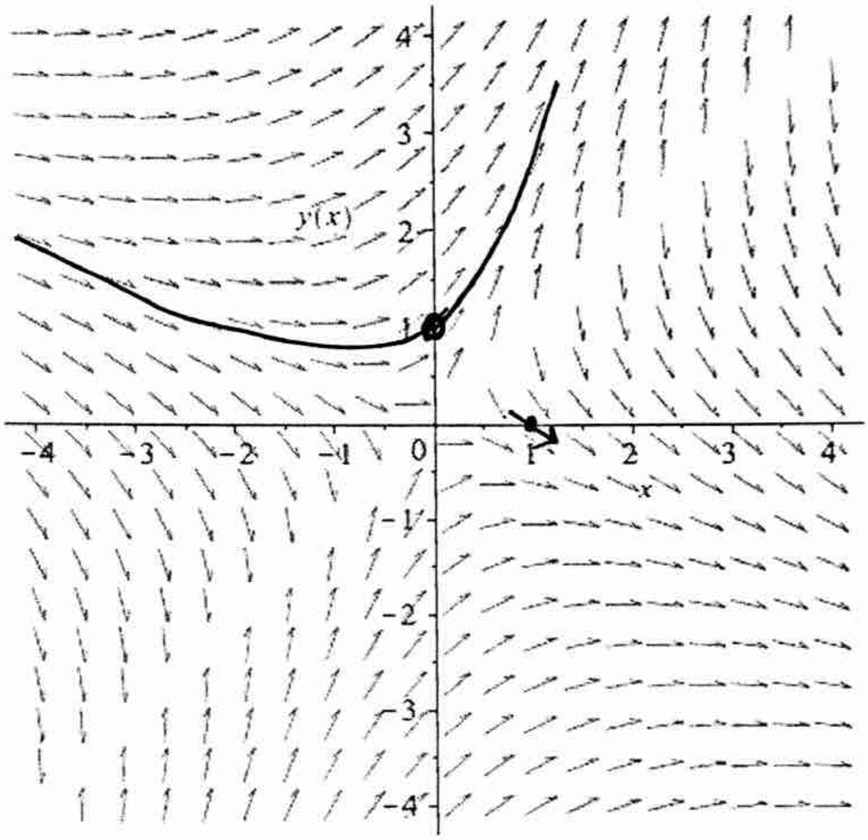
Then $|f'''(x)| \le 27$ and we may use $M_3 = 27$.

Then for $0 \le x \le 0.1$, we have

$$|R_2(x)| \le \frac{M_3 |x-0|^3}{3!} \le \boxed{\frac{27(.1)^3}{6}}$$

Note $M_3 = 27\sin(.3)$ is also valid.

10. Consider the slope field pictured below.



(a) (3 points) Which of the differential equations below matches this slope field?

(a) $\frac{dy}{dx} = \frac{x}{y}$, (b) $\frac{dy}{dx} = x^2$, $\boxed{\text{(c) } \frac{dy}{dx} = \frac{y+x}{y-x},}$ (d) $\frac{dy}{dx} = \sin y$.

At (1,0) $\quad \infty \quad 1 \quad -1 \quad 0$

(b) (3 points) Sketch the solution to this differential equation that satisfies $y(0) = 1$ on the slope field.

11. Solve the following differential equations. Either give the general solution, or solve for a particular solution satisfying the given initial conditions. Your solution must give an explicit formula for y for full credit.

(a) (8 points) $\frac{dy}{dx} = x\sqrt{1-y^2}$, $y(0) = 1$.

separable

$$\frac{dy}{\sqrt{1-y^2}} = x\,dx$$

$$\int \frac{dy}{\sqrt{1-y^2}} = \int x\,dx$$

$$\arcsin(y) = \frac{1}{2}x^2 + C$$

$$y = \sin\left(\frac{1}{2}x^2 + C\right)$$

Solve for C: $x=0, y=1$

$$1 = \sin(0 + C)$$

$C = \frac{\pi}{2}$ works

$$y = \sin\left(\frac{1}{2}x^2 + \frac{\pi}{2}\right).$$

(b) (8 points) $y' + \frac{1}{x}y = \cos x$. 1^{st} order linear

$\int \frac{1}{x}\,dx = \ln|x|$

Integrating factor is $v(x) = e^{\ln|x|} = |x|$.

Choose to use $v(x) = x$.

$x y' + x \cdot \frac{1}{x} y = x \cos x$

$\frac{d}{dx}(xy) = x \cos x$

$xy = \int x \cos x \, dx$

$u = x \quad dv = \cos x \, dx$

$du = dx \quad v = \sin x$

$= x \sin x - \int \sin x \, dx$

$xy = x \sin x + \cos x + C$

$y = \frac{1}{x}(x \sin x + \cos x + C)$

$y = \sin x + \frac{\cos x}{x} + \frac{C}{x}$