Name: Solutions Section: 7 8

Determine if the following series converge or diverge. You may use the techniques of geometric series, telescoping series, the divergence test, p-series, and the integral test. Show your work and clearly state which test you are using.

1. $\sum_{n=1}^{\infty} \frac{3}{n^5}$ $= 3 \cdot \sum_{n=1}^{\infty} \frac{1}{n^5}$ p-series

$p = 5 > 1$

series converges.

2. $\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$ Divergence test!

$\lim_{n\to\infty} \frac{n-1}{2n+1} \overset{H}{=} \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0$

series diverges

3. $\sum_{n=1}^{\infty} \frac{2^n}{7^{n-2}}$ Geometric series

$\frac{2^n}{7^{n-2}} = \frac{2^n}{7^n \cdot 7^{-2}} = 7 \cdot \left(\frac{2}{7}\right)^n$ $|\text{ratio}| = \left|\frac{2}{7}\right| < 1$

series converges

4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$ $= \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ p-series

$p = 1/3 \leq 1$

series diverges

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

Integral test: $f(x) = \dfrac{1}{1+x^2}$

f is positive, cts, and decreasing

$$\int_1^{\infty} \frac{1}{1+x^2}\,dx = \lim_{n\to\infty} \int_1^{t} \frac{1}{1+x^2}\,dx = \lim_{n\to\infty} \Big[\arctan x\Big]_1^{t}$$

$$= \lim_{n\to\infty} (\arctan t - \arctan 1)$$

$$= \pi/2 - \pi/4 = \pi/4$$

$\therefore$ series converges

6. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{3}{n}\right)^n$

Divergence test

$\left(1+\frac{3}{n}\right)^n \longrightarrow e^3 \neq 0$ series diverges

7. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$ Telescoping series

$$S_n = \left(\frac{1}{\sqrt{1}} - \cancel{\frac{1}{\sqrt{2}}}\right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \cancel{\frac{1}{\sqrt{3}}}\right) + \left(\cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{4}}}\right) + \cdots + \left(\cancel{\frac{1}{\sqrt{n}}} - \frac{1}{\sqrt{n+1}}\right)$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \lim_{n\to\infty}\left(1 - \frac{1}{\sqrt{n+1}}\right) = 1$$ series converges

8. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$

Integral test: $f(x) = \dfrac{1}{x\ln x}$ is pos, cts, dec.

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x\ln x}\,dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\,du$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$u(2) = \ln 2$

$u(t) = \ln t$

$$= \lim_{t\to\infty} \ln|u| \Big]_{\ln 2}^{\ln t}$$

$$= \lim_{t\to\infty} \big(\ln|\ln t| - \ln|\ln 2|\big)$$

$$= \infty$$

series diverges