(b) Estimate the rates of population growth in 1800 and 1850 by averaging slopes of secant lines.
(c) Use the exponential model in part (a) to estimate the rates of growth in 1800 and 1850. Compare these estimates with the ones in part (b).
(d) Use the exponential model to predict the population in 1870. Compare with the actual population of 38,558,000. Can you explain the discrepancy?

71–82 Evaluate the integral.

71. $\int_2^4 \frac{3}{x}\,dx$

72. $\int_0^3 \frac{dx}{5x+1}$

73. $\int_1^2 \frac{dt}{8-3t}$

74. $\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

75. $\int_1^e \frac{x^2+x+1}{x}\,dx$

76. $\int \frac{\cos(\ln t)}{t}\,dt$

77. $\int \frac{(\ln x)^2}{x}\,dx$

78. $\int \frac{\cos x}{2+\sin x}\,dx$

79. $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

80. $\int \frac{e^x}{e^x+1}\,dx$

81. $\int_0^4 2^s\,ds$

82. $\int x2^{x^2}\,dx$

---

83. Show that $\int \cot x\,dx = \ln|\sin x| + C$ by (a) differentiating the right side of the equation and (b) using the method of Example 11.

84. Sketch the region enclosed by the curves

$$y = \frac{\ln x}{x} \quad \text{and} \quad y = \frac{(\ln x)^2}{x}$$

and find its area.

85. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{\sqrt{x+1}}$$

from 0 to 1 about the x-axis.

86. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{x^2+1}$$

from 0 to 3 about the y-axis.

87. The work done by a gas when it expands from volume V_1 to volume V_2 is $W = \int_{V_1}^{V_2} P\,dV$, where $P = P(V)$ is the pressure as a function of the volume V. (See Exercise 5.4.29.) Boyle's Law states that when a quantity of gas expands at constant temperature, $PV = C$, where C is a constant. If the initial volume is 600 cm^3 and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm^3.

88. Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$.

89. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

90. If $f(x) = e^x + \ln x$ and $h(x) = f^{-1}(x)$, find $h'(e)$.

91. For what values of m do the line $y = mx$ and the curve $y = x/(x^2+1)$ enclose a region? Find the area of the region.

92. (a) Find the linear approximation to $f(x) = \ln x$ near 1.
(b) Illustrate part (a) by graphing f and its linearization.
(c) For what values of x is the linear approximation accurate to within 0.1?

93. Use the definition of derivative to prove that

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

94. Show that $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ for any $x > 0$.

6.2* The Natural Logarithmic Function

If your instructor has assigned Sections 6.2–6.4 (pp. 408–438), you need not read Sections 6.2*, 6.3*, and 6.4* (pp. 438–465).

In this section we define the natural logarithm as an integral and then show that it obeys the usual laws of logarithms. The Fundamental Theorem makes it easy to differentiate this function.

1 Definition The **natural logarithmic function** is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

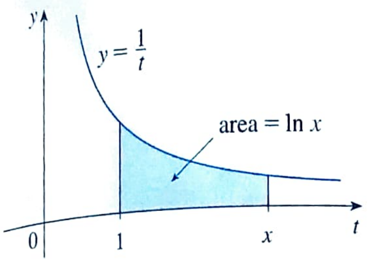

FIGURE 1

The existence of this function depends on the fact that the integral of a continuous function always exists. If $x > 1$, then $\ln x$ can be interpreted geometrically as the area under the hyperbola $y = 1/t$ from $t = 1$ to $t = x$. (See Figure 1.) For $x = 1$, we have

$$\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$$

For $0 < x < 1$,

$$\ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$$

and so $\ln x$ is the negative of the area shaded in Figure 2.

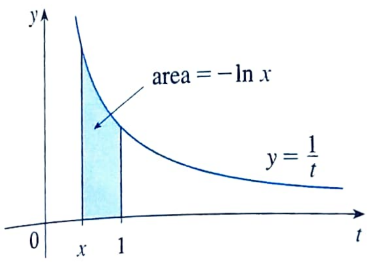

FIGURE 2

EXAMPLE 1

(a) By comparing areas, show that $\frac{1}{2} < \ln 2 < \frac{3}{4}$.
(b) Use the Midpoint Rule with $n = 10$ to estimate the value of $\ln 2$.

SOLUTION

(a) We can interpret $\ln 2$ as the area under the curve $y = 1/t$ from 1 to 2. From Figure 3 we see that this area is larger than the area of rectangle $BCDE$ and smaller than the area of trapezoid $ABCD$. Thus we have

$$\tfrac{1}{2} \cdot 1 < \ln 2 < 1 \cdot \tfrac{1}{2}\left(1 + \tfrac{1}{2}\right)$$

$$\tfrac{1}{2} < \ln 2 < \tfrac{3}{4}$$

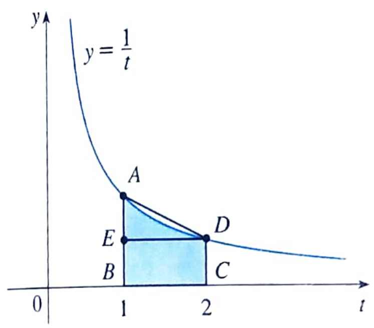

FIGURE 3

(b) If we use the Midpoint Rule with $f(t) = 1/t$, $n = 10$, and $\Delta t = 0.1$, we get

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt \approx (0.1)[f(1.05) + f(1.15) + \cdots + f(1.95)]$$

$$= (0.1)\left(\frac{1}{1.05} + \frac{1}{1.15} + \cdots + \frac{1}{1.95}\right) \approx 0.693$$ ■

Notice that the integral that defines $\ln x$ is exactly the type of integral discussed in Part 1 of the Fundamental Theorem of Calculus (see Section 4.3). In fact, using that theorem, we have

$$\frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}$$

and so

2
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

We now use this differentiation rule to prove the following properties of the logarithm function.

3 Laws of Logarithms If x and y are positive numbers and r is a rational number, then

1. $\ln(xy) = \ln x + \ln y$ **2.** $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ **3.** $\ln(x^r) = r \ln x$

PROOF

1. Let $f(x) = \ln(ax)$, where a is a positive constant. Then, using Equation 2 and the Chain Rule, we have

$$f'(x) = \frac{1}{ax}\frac{d}{dx}(ax) = \frac{1}{ax} \cdot a = \frac{1}{x}$$

Therefore $f(x)$ and $\ln x$ have the same derivative and so they must differ by a constant:

$$\ln(ax) = \ln x + C$$

Putting $x = 1$ in this equation, we get $\ln a = \ln 1 + C = 0 + C = C$. Thus

$$\ln(ax) = \ln x + \ln a$$

If we now replace the constant a by any number y, we have

$$\ln(xy) = \ln x + \ln y$$

2. Using Law 1 with $x = 1/y$, we have

$$\ln\frac{1}{y} + \ln y = \ln\left(\frac{1}{y} \cdot y\right) = \ln 1 = 0$$

and so
$$\ln\frac{1}{y} = -\ln y$$

Using Law 1 again, we have

$$\ln\left(\frac{x}{y}\right) = \ln\left(x \cdot \frac{1}{y}\right) = \ln x + \ln\frac{1}{y} = \ln x - \ln y$$

The proof of Law 3 is left as an exercise.

EXAMPLE 2 Expand the expression $\ln\dfrac{(x^2+5)^4 \sin x}{x^3+1}$.

SOLUTION Using Laws 1, 2, and 3, we get

$$\ln\frac{(x^2+5)^4 \sin x}{x^3+1} = \ln(x^2+5)^4 + \ln \sin x - \ln(x^3+1)$$
$$= 4\ln(x^2+5) + \ln \sin x - \ln(x^3+1)$$

EXAMPLE 3 Express $\ln a + \frac{1}{2}\ln b$ as a single logarithm.

SOLUTION Using Laws 3 and 1 of logarithms, we have

$$\ln a + \tfrac{1}{2}\ln b = \ln a + \ln b^{1/2}$$
$$= \ln a + \ln\sqrt{b}$$
$$= \ln\left(a\sqrt{b}\right)$$

In order to graph $y = \ln x$, we first determine its limits:

4 (a) $\displaystyle\lim_{x\to\infty} \ln x = \infty$ (b) $\displaystyle\lim_{x\to 0^+} \ln x = -\infty$

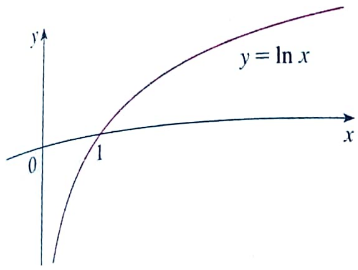

FIGURE 4

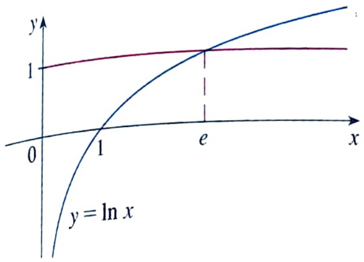

FIGURE 5

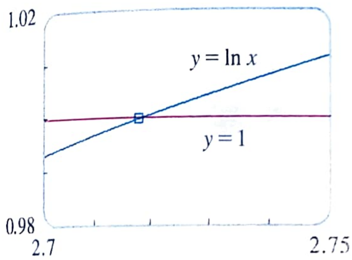

FIGURE 6

PROOF

(a) Using Law 3 with $x = 2$ and $r = n$ (where n is any positive integer), we have $\ln(2^n) = n \ln 2$. Now $\ln 2 > 0$, so this shows that $\ln(2^n) \to \infty$ as $n \to \infty$. But $\ln x$ is an increasing function since its derivative $1/x > 0$. Therefore $\ln x \to \infty$ as $x \to \infty$.

(b) If we let $t = 1/x$, then $t \to \infty$ as $x \to 0^+$. Thus, using (a), we have

$$\lim_{x \to 0^+} \ln x = \lim_{t \to \infty} \ln\left(\frac{1}{t}\right) = \lim_{t \to \infty} (-\ln t) = -\infty$$ ■

If $y = \ln x$, $x > 0$, then

$$\frac{dy}{dx} = \frac{1}{x} > 0 \quad \text{and} \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0$$

which shows that $\ln x$ is increasing and concave downward on $(0, \infty)$. Putting this information together with (4), we draw the graph of $y = \ln x$ in Figure 4.

Since $\ln 1 = 0$ and $\ln x$ is an increasing continuous function that takes on arbitrarily large values, the Intermediate Value Theorem shows that there is a number where $\ln x$ takes on the value 1. (See Figure 5.) This important number is denoted by e.

5 Definition e is the number such that $\ln e = 1$.

EXAMPLE 4 Use a graphing calculator or computer to estimate the value of e.

SOLUTION According to Definition 5, we estimate the value of e by graphing the curves $y = \ln x$ and $y = 1$ and determining the x-coordinate of the point of intersection. By zooming in repeatedly, as in Figure 6, we find that

$$e \approx 2.718$$ ■

With more sophisticated methods, it can be shown that the approximate value of e, to 20 decimal places, is

$$e \approx 2.71828182845904523536$$

The decimal expansion of e is nonrepeating because e is an irrational number.

Now let's use Formula 2 to differentiate functions that involve the natural logarithmic function.

EXAMPLE 5 Differentiate $y = \ln(x^3 + 1)$.

SOLUTION To use the Chain Rule, we let $u = x^3 + 1$. Then $y = \ln u$, so

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{u}\frac{du}{dx}$$

$$= \frac{1}{x^3 + 1}(3x^2) = \frac{3x^2}{x^3 + 1}$$ ■

In general, if we combine Formula 2 with the Chain Rule as in Example 5, we get

6 $$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

EXAMPLE 6 Find $\dfrac{d}{dx}\ln(\sin x)$.

SOLUTION Using (6), we have

$$\frac{d}{dx}\ln(\sin x) = \frac{1}{\sin x}\frac{d}{dx}(\sin x) = \frac{1}{\sin x}\cos x = \cot x$$

EXAMPLE 7 Differentiate $f(x) = \sqrt{\ln x}$.

SOLUTION This time the logarithm is the inner function, so the Chain Rule gives

$$f'(x) = \tfrac{1}{2}(\ln x)^{-1/2}\frac{d}{dx}(\ln x) = \frac{1}{2\sqrt{\ln x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\ln x}}$$

EXAMPLE 8 Find $\dfrac{d}{dx}\ln\dfrac{x+1}{\sqrt{x-2}}$.

SOLUTION 1

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}} = \frac{1}{\dfrac{x+1}{\sqrt{x-2}}}\frac{d}{dx}\frac{x+1}{\sqrt{x-2}}$$

$$= \frac{\sqrt{x-2}}{x+1}\,\frac{\sqrt{x-2}\cdot 1 - (x+1)\left(\tfrac{1}{2}\right)(x-2)^{-1/2}}{x-2}$$

$$= \frac{x - 2 - \tfrac{1}{2}(x+1)}{(x+1)(x-2)}$$

$$= \frac{x-5}{2(x+1)(x-2)}$$

Figure 7 shows the graph of the function f of Example 8 together with the graph of its derivative. It gives a visual check on our calculation. Notice that $f'(x)$ is large negative when f is rapidly decreasing and $f'(x) = 0$ when f has a minimum.

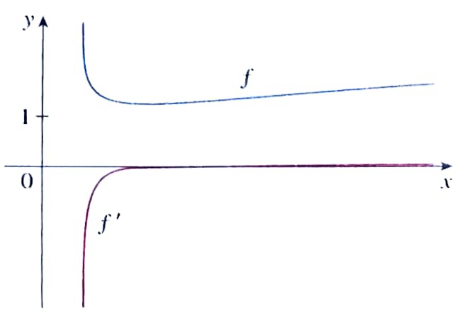

FIGURE 7

SOLUTION 2 If we first simplify the given function using the Laws of Logarithms, then the differentiation becomes easier:

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x+2}} = \frac{d}{dx}\left[\ln(x+1) - \tfrac{1}{2}\ln(x-2)\right]$$

$$= \frac{1}{x+1} - \frac{1}{2}\left(\frac{1}{x-2}\right)$$

(This answer can be left as written, but if we used a common denominator we would see that it gives the same answer as in Solution 1.)

EXAMPLE 9 Discuss the curve $y = \ln(4 - x^2)$ using the guidelines of Section 3.5.

A. The domain is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\} = (-2, 2)$$

B. The y-intercept is $f(0) = \ln 4$. To find the x-intercept we set

$$y = \ln(4 - x^2) = 0$$

We know that $\ln 1 = 0$, so we have $4 - x^2 = 1 \Rightarrow x^2 = 3$ and therefore the x-intercepts are $\pm\sqrt{3}$.

C. Since $f(-x) = f(x)$, f is even and the curve is symmetric about the y-axis.

D. We look for vertical asymptotes at the endpoints of the domain. Since $4 - x^2 \to 0^+$ as $x \to 2^-$ and also as $x \to -2^+$, we have

$$\lim_{x\to 2^-} \ln(4 - x^2) = -\infty \qquad \lim_{x\to -2^+} \ln(4 - x^2) = -\infty$$

Thus the lines $x = 2$ and $x = -2$ are vertical asymptotes.

E.
$$f'(x) = \frac{-2x}{4 - x^2}$$

Since $f'(x) > 0$ when $-2 < x < 0$ and $f'(x) < 0$ when $0 < x < 2$, f is increasing on $(-2, 0)$ and decreasing on $(0, 2)$.

F. The only critical number is $x = 0$. Since f' changes from positive to negative at 0, $f(0) = \ln 4$ is a local maximum by the First Derivative Test.

G.
$$f''(x) = \frac{(4 - x^2)(-2) + 2x(-2x)}{(4 - x^2)^2} = \frac{-8 - 2x^2}{(4 - x^2)^2}$$

Since $f''(x) < 0$ for all x, the curve is concave downward on $(-2, 2)$ and has no inflection point.

H. Using this information, we sketch the curve in Figure 8. ■

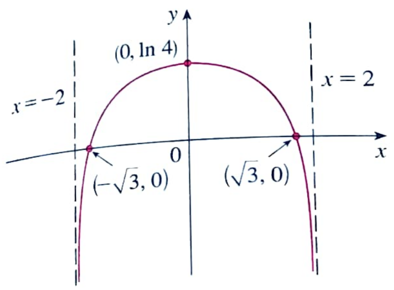

FIGURE 8
$y = \ln(4 - x^2)$

EXAMPLE 10 Find $f'(x)$ if $f(x) = \ln|x|$.

SOLUTION Since

$$f(x) = \begin{cases} \ln x & \text{if } x > 0 \\ \ln(-x) & \text{if } x < 0 \end{cases}$$

it follows that

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\[2ex] \dfrac{1}{-x}(-1) = \dfrac{1}{x} & \text{if } x < 0 \end{cases}$$

Thus $f'(x) = 1/x$ for all $x \neq 0$. ■

Figure 9 shows the graph of the function $f(x) = \ln|x|$ in Example 10 and its derivative $f'(x) = 1/x$. Notice that when x is small, the graph of $y = \ln|x|$ is steep and so $f'(x)$ is large (positive or negative).

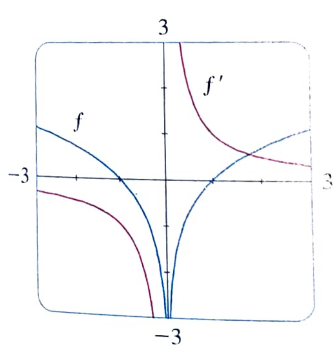

FIGURE 9

The result of Example 10 is worth remembering:

7
$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

The corresponding integration formula is

8
$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Notice that this fills the gap in the rule for integrating power functions:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C \qquad \text{if } n \neq -1$$

The missing case ($n = -1$) is supplied by Formula 8.

EXAMPLE 11 Evaluate $\displaystyle\int \frac{x}{x^2+1} \, dx$.

SOLUTION We make the substitution $u = x^2 + 1$ because the differential $du = 2x\,dx$ occurs (except for the constant factor 2). Thus $x\,dx = \frac{1}{2}\,du$ and

$$\int \frac{x}{x^2+1} \, dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|x^2+1| + C = \frac{1}{2}\ln(x^2+1) + C$$

Notice that we removed the absolute value signs because $x^2 + 1 > 0$ for all x. We could use the Laws of Logarithms to write the answer as

$$\ln\sqrt{x^2+1} + C$$

but this isn't necessary.

Since the function $f(x) = (\ln x)/x$ in Example 12 is positive for $x > 1$, the integral represents the area of the shaded region in Figure 10.

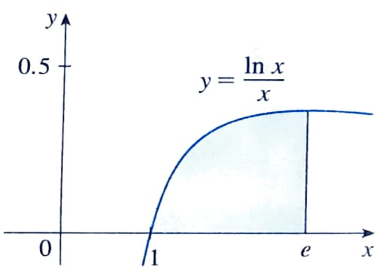

FIGURE 10

EXAMPLE 12 Calculate $\displaystyle\int_1^e \frac{\ln x}{x} \, dx$.

SOLUTION We let $u = \ln x$ because its differential $du = dx/x$ occurs in the integral. When $x = 1$, $u = \ln 1 = 0$; when $x = e$, $u = \ln e = 1$. Thus

$$\int_1^e \frac{\ln x}{x} \, dx = \int_0^1 u \, du = \frac{u^2}{2}\Big]_0^1 = \frac{1}{2}$$

EXAMPLE 13 Calculate $\displaystyle\int \tan x \, dx$.

SOLUTION First we write tangent in terms of sine and cosine:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$$

This suggests that we should substitute $u = \cos x$, since then $du = -\sin x \, dx$ and so $\sin x \, dx = -du$:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{1}{u} \, du$$

$$= -\ln|u| + C = -\ln|\cos x| + C$$

Since $-\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln(1/|\cos x|) = \ln|\sec x|$, the result of Example 13 can also be written as

$$\boxed{9} \qquad \int \tan x \, dx = \ln|\sec x| + C$$

Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. The method used in the following example is called **logarithmic differentiation**.

EXAMPLE 14 Differentiate $y = \dfrac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}$.

SOLUTION We take logarithms of both sides of the equation and use the Laws of Logarithms to simplify:

$$\ln y = \tfrac{3}{4} \ln x + \tfrac{1}{2} \ln(x^2 + 1) - 5 \ln(3x + 2)$$

Differentiating implicitly with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 + 1} - 5 \cdot \frac{3}{3x + 2}$$

Solving for dy/dx, we get

$$\frac{dy}{dx} = y\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$

If we hadn't used logarithmic differentiation in Example 14, we would have had to use both the Quotient Rule and the Product Rule. The resulting calculation would have been horrendous.

Because we have an explicit expression for y, we can substitute and write

$$\frac{dy}{dx} = \frac{x^{3/4}\sqrt{x^2 + 1}}{(3x + 2)^5}\left(\frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x + 2}\right)$$ ■

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined, but we can still use logarithmic differentiation by first writing $|y| = |f(x)|$ and then using Equation 7.

6.2* EXERCISES

1–4 Use the Laws of Logarithms to expand the quantity.

1. $\ln \sqrt{ab}$
2. $\ln \sqrt[3]{\dfrac{x - 1}{x + 1}}$
3. $\ln \dfrac{x^2}{y^3 z^4}$
4. $\ln\left(s^4 \sqrt{t\sqrt{u}}\right)$

5–10 Express the quantity as a single logarithm.

5. $2 \ln x + 3 \ln y - \ln z$
6. $\log_{10} 4 + \log_{10} a - \frac{1}{3} \log_{10}(a + 1)$
7. $\ln 10 + 2 \ln 5$
8. $\ln 3 + \frac{1}{3} \ln 8$
9. $\frac{1}{3} \ln(x + 2)^3 + \frac{1}{2}[\ln x - \ln(x^2 + 3x + 2)^2]$
10. $\ln b + 2 \ln c - 3 \ln d$

11–14 Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graph given in Figure 4 and, if necessary, the transformations of Section 1.3.

11. $y = -\ln x$
12. $y = \ln |x|$
13. $y = \ln(x + 3)$
14. $y = 1 + \ln(x - 2)$

15–16 Find the limit.

15. $\lim_{x \to 3^+} \ln(x^2 - 9)$

16. $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

17–36 Differentiate the function.

17. $f(x) = x^3 \ln x$

18. $f(x) = x \ln x - x$

19. $f(x) = \sin(\ln x)$

20. $f(x) = \ln(\sin^2 x)$

21. $f(x) = \ln \dfrac{1}{x}$

22. $y = \dfrac{1}{\ln x}$

23. $f(x) = \sin x \ln(5x)$

24. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$

25. $g(x) = \ln \dfrac{a - x}{a + x}$

26. $g(t) = \sqrt{1 + \ln t}$

27. $G(y) = \ln \dfrac{(2y + 1)^5}{\sqrt{y^2 + 1}}$

28. $H(z) = \ln \sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

29. $F(t) = (\ln t)^2 \sin t$

30. $P(v) = \dfrac{\ln v}{1 - v}$

31. $f(u) = \dfrac{\ln u}{1 + \ln(2u)}$

32. $y = (\ln \tan x)^2$

33. $y = \ln|2 - x - 5x^2|$

34. $y = \ln \tan^2 x$

35. $y = \tan[\ln(ax + b)]$

36. $y = \ln(\csc x - \cot x)$

37–38 Find y' and y''.

37. $y = \sqrt{x} \ln x$

38. $y = \ln(1 + \ln x)$

39–42 Differentiate f and find the domain of f.

39. $f(x) = \dfrac{x}{1 - \ln(x - 1)}$

40. $f(x) = \ln(x^2 - 2x)$

41. $f(x) = \sqrt{1 - \ln x}$

42. $f(x) = \ln \ln \ln x$

43. If $f(x) = \ln(x + \ln x)$, find $f'(1)$.

44. If $f(x) = \dfrac{\ln x}{x}$, find $f''(e)$.

45–46 Find $f'(x)$. Check that your answer is reasonable by comparing the graphs of f and f'.

45. $f(x) = \sin x + \ln x$

46. $f(x) = \ln(x^2 + x + 1)$

47–48 Find an equation of the tangent line to the curve at the given point.

47. $y = \sin(2 \ln x)$, $(1, 0)$

48. $y = \ln(x^3 - 7)$, $(2, 0)$

49. Find y' if $y = \ln(x^2 + y^2)$.

50. Find y' if $\ln xy = y \sin x$.

51. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

52. Find $\dfrac{d^9}{dx^9}(x^8 \ln x)$.

53–54 Use a graph to estimate the roots of the equation correct to one decimal place. Then use these estimates as the initial approximations in Newton's method to find the roots correct to six decimal places.

53. $(x - 4)^2 = \ln x$

54. $\ln(4 - x^2) = x$

55–58 Discuss the curve under the guidelines of Section 3.5.

55. $y = \ln(\sin x)$

56. $y = \ln(\tan^2 x)$

57. $y = \ln(1 + x^2)$

58. $y = \ln(1 + x^3)$

CAS **59.** If $f(x) = \ln(2x + x \sin x)$, use the graphs of f, f', and f'' to estimate the intervals of increase and the inflection points of f on the interval $(0, 15]$.

60. Investigate the family of curves $f(x) = \ln(x^2 + c)$. What happens to the inflection points and asymptotes as c changes? Graph several members of the family to illustrate what you discover.

61–64 Use logarithmic differentiation to find the derivative of the function.

61. $y = (x^2 + 2)^2(x^4 + 4)^4$

62. $y = \dfrac{(x + 1)^4(x - 5)^3}{(x - 3)^8}$

63. $y = \sqrt{\dfrac{x - 1}{x^4 + 1}}$

64. $y = \dfrac{(x^3 + 1)^4 \sin^2 x}{x^{1/3}}$

65–74 Evaluate the integral.

65. $\displaystyle\int_2^4 \frac{3}{x}\, dx$

66. $\displaystyle\int_0^3 \frac{dx}{5x + 1}$

67. $\displaystyle\int_1^2 \frac{dt}{8 - 3t}$

68. $\displaystyle\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

69. $\displaystyle\int_1^e \frac{x^2 + x + 1}{x}\, dx$

70. $\displaystyle\int_e^6 \frac{dx}{x \ln x}$

71. $\displaystyle\int \frac{(\ln x)^2}{x}\, dx$

72. $\displaystyle\int \frac{\cos x}{2 + \sin x}\, dx$

73. $\displaystyle\int \frac{\sin 2x}{1 + \cos^2 x}\, dx$

74. $\displaystyle\int \frac{\cos(\ln t)}{t}\, dt$

75. Show that $\int \cot x\, dx = \ln|\sin x| + C$ by (a) differentiating the right side of the equation and (b) using the method of Example 13.

76. Sketch the region enclosed by the curves

$$y = \frac{\ln x}{x} \quad \text{and} \quad y = \frac{(\ln x)^2}{x}$$

and find its area.

77. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{\sqrt{x+1}}$$

from 0 to 1 about the x-axis.

78. Find the volume of the solid obtained by rotating the region under the curve

$$y = \frac{1}{x^2 + 1}$$

from 0 to 3 about the y-axis.

79. The work done by a gas when it expands from volume V_1 to volume V_2 is $W = \int_{V_1}^{V_2} P\,dV$, where $P = P(V)$ is the pressure as a function of the volume V. (See Exercise 5.4.29.) Boyle's Law states that when a quantity of gas expands at constant temperature, $PV = C$, where C is a constant. If the initial volume is 600 cm^3 and the initial pressure is 150 kPa, find the work done by the gas when it expands at constant temperature to 1000 cm^3.

80. Find f if $f''(x) = x^{-2}$, $x > 0$, $f(1) = 0$, and $f(2) = 0$.

81. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

82. (a) Find the linear approximation to $f(x) = \ln x$ near 1.
(b) Illustrate part (a) by graphing f and its linearization.
(c) For what values [illegible] the linear approximation accurate to with[illegible]

83. (a) By comparing areas, show that

$$\tfrac{1}{3} < \ln 1.5 < \tfrac{5}{12}$$

(b) Use the Midpoint Rule with $n = 10$ to estimate $\ln 1.5$.

84. Refer to Example 1.
(a) Find an equation of the tangent line to the curve $y = 1/t$ that is parallel to the secant line AD.
(b) Use part (a) to show that $\ln 2 > 0.66$.

85. By comparing areas, show that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}$$

86. Prove the third law of logarithms. [*Hint:* Start by showing that both sides of the equation have the same derivative.]

87. For what values of m do the line $y = mx$ and the curve $y = x/(x^2 + 1)$ enclose a region? Find the area of the region.

88. (a) Compare the rates of growth of $f(x) = x^{0.1}$ and $g(x) = \ln x$ by graphing both f and g in several viewing rectangles. When does the graph of f finally surpass the graph of g?
(b) Graph the function $h(x) = (\ln x)/x^{0.1}$ in a viewing rectangle that displays the behavior of the function as $x \to \infty$.
(c) Find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \frac{\ln x}{x^{0.1}} < 0.1$$

89. Use the definition of derivative to prove that

$$\lim_{x \to 0} \frac{\ln(1 + x)}{x} = 1$$

6.3* The Natural Exponential Function

Since ln is an increasing function, it is one-to-one and therefore has an inverse function, which we denote by exp. Thus, according to the definition of an inverse function,

$f^{-1}(x) = y \iff f(y) = x$

1
$$\exp(x) = y \iff \ln y = x$$

and the cancellation equations are

$f^{-1}(f(x)) = x$
$f(f^{-1}(x)) = x$

2
$$\exp(\ln x) = x \quad \text{and} \quad \ln(\exp x) = x$$

In particular, we have

$$\exp(0) = 1 \quad \text{since} \quad \ln 1 = 0$$

$$\exp(1) = e \quad \text{since} \quad \ln e = 1$$