15.2 Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval. But for double integrals, we want to be able to integrate a function f not just over rectangles but also over regions D of more general shape, such as the one illustrated in Figure 1. We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2. Then we define a new function F with domain R by

1
$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

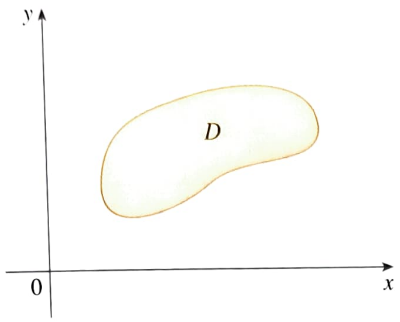


FIGURE 1

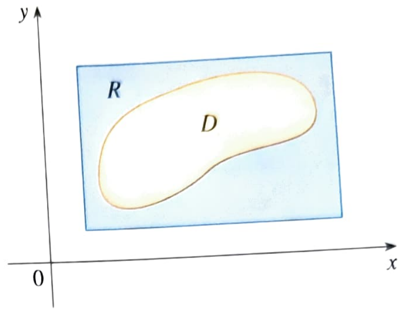


FIGURE 2

If F is integrable over R, then we define the **double integral of f over D** by

2
$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

Definition 2 makes sense because R is a rectangle and so $\iint_R F(x, y)\, dA$ has been previously defined in Section 15.1. The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside D and so they contribute nothing to the integral. This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geqslant 0$, we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x, y)$ (the graph of f). You can see that this is reasonable by comparing the graphs of f and F in Figures 3 and 4 and remembering that $\iint_R F(x, y)\, dA$ is the volume under the graph of F.

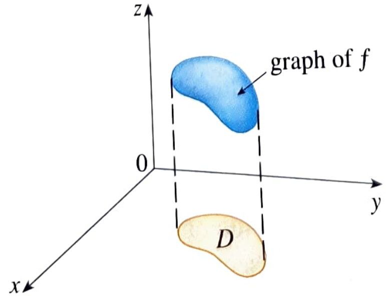


FIGURE 3

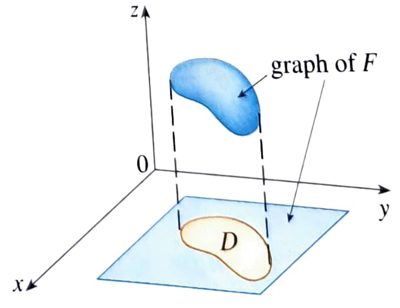


FIGURE 4

Figure 4 also shows that F is likely to have discontinuities at the boundary poin of D. Nonetheless, if f is continuous on D and the boundary curve of D is "well behave (in a sense outside the scope of this book), then it can be shown that $\iint_R F(x, y)\, dA$ exis

EXAMPLE 5 Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$.

SOLUTION If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int \sin(y^2)\,dy$. But it's impossible to do so in finite terms since $\int \sin(y^2)\,dy$ is not an elementary function. (See the end of Section 7.5.) So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. Using (3) backward, we have

$$\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx = \iint_D \sin(y^2)\,dA$$

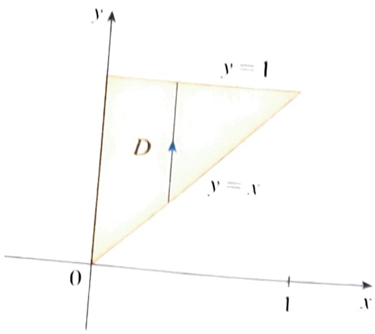


FIGURE 15
D as a type I region

where $$D = \{(x, y) \mid 0 \leqslant x \leqslant 1,\ x \leqslant y \leqslant 1\}$$

We sketch this region D in Figure 15. Then from Figure 16 we see that an alternative description of D is

$$D = \{(x, y) \mid 0 \leqslant y \leqslant 1,\ 0 \leqslant x \leqslant y\}$$

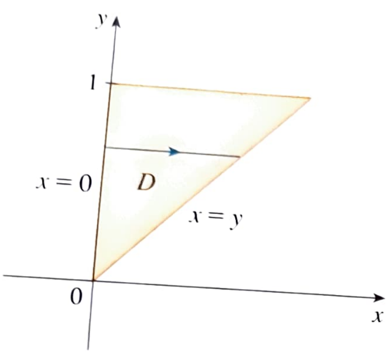


FIGURE 16
D as a type II region

This enables us to use (5) to express the double integral as an iterated integral in the reverse order:

$$\begin{aligned}\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx &= \iint_D \sin(y^2)\,dA \\ &= \int_0^1 \int_0^y \sin(y^2)\,dx\,dy = \int_0^1 \Big[x\sin(y^2)\Big]_{x=0}^{x=y} dy \\ &= \int_0^1 y\sin(y^2)\,dy = -\tfrac{1}{2}\cos(y^2)\Big]_0^1 = \tfrac{1}{2}(1 - \cos 1)\end{aligned}$$ ■

Properties of Double Integrals

We assume that all of the following integrals exist. For rectangular regions D the first three properties can be proved in the same manner as in Section 4.2. And then for general regions the properties follow from Definition 2.

6 $$\iint_D [f(x, y) + g(x, y)]\,dA = \iint_D f(x, y)\,dA + \iint_D g(x, y)\,dA$$

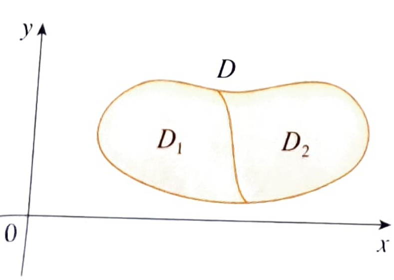
7 $$\iint_D cf(x, y)\,dA = c\iint_D f(x, y)\,dA \qquad \text{where } c \text{ is a constant}$$

If $f(x, y) \geqslant g(x, y)$ for all (x, y) in D, then

8 $$\iint_D f(x, y)\,dA \geqslant \iint_D g(x, y)\,dA$$

The next property of double integrals is similar to the property of single integrals given by the equation $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then



URE 17

9 $$\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA$$

Property 9 can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure. (See Exercises 57 and 58.)

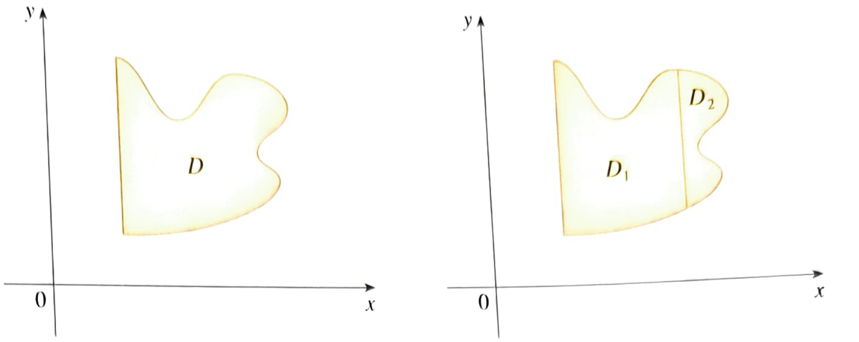


FIGURE 18

(a) D is neither type I nor type II.

(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

10
$$\iint_D 1\, dA = A(D)$$

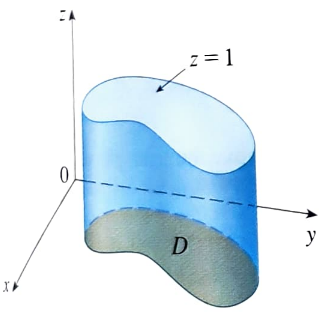


FIGURE 19
Cylinder with base D and height 1

Figure 19 illustrates why Equation 10 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1\, dA$.

Finally, we can combine Properties 7, 8, and 10 to prove the following property. (See Exercise 63.)

11 If $m \leq f(x, y) \leq M$ for all (x, y) in D, then

$$mA(D) \leq \iint_D f(x, y)\, dA \leq MA(D)$$

EXAMPLE 6 Use Property 11 to estimate the integral $\iint_D e^{\sin x \cos y}\, dA$, where D is the disk with center the origin and radius 2.

SOLUTION Since $-1 \leq \sin x \leq 1$ and $-1 \leq \cos y \leq 1$, we have $-1 \leq \sin x \cos y \leq 1$ and therefore

$$e^{-1} \leq e^{\sin x \cos y} \leq e^1 = e$$

Thus, using $m = e^{-1} = 1/e$, $M = e$, and $A(D) = \pi(2)^2$ in Property 11, we obtain

$$\frac{4\pi}{e} \leq \iint_D e^{\sin x \cos y}\, dA \leq 4\pi e$$ ■

15.2 EXERCISES

1–6 Evaluate the iterated integral.

1. $\int_1^5 \int_0^x (8x - 2y)\, dy\, dx$

2. $\int_0^2 \int_0^{y^2} x^2 y\, dx\, dy$

3. $\int_0^1 \int_0^y x e^{y^3}\, dx\, dy$

4. $\int_0^{\pi/2} \int_0^x x \sin y\, dy\, dx$

5. $\int_0^1 \int_0^{s^2} \cos(s^3)\, dt\, ds$

6. $\int_0^1 \int_0^{e^v} \sqrt{1 + e^v}\, dw\, dv$

7–10 Evaluate the double integral.

7. $\iint_D \frac{y}{x^2 + 1}\, dA$, $D = \{(x, y) \mid 0 \leqslant x \leqslant 4, 0 \leqslant y \leqslant \sqrt{x}\}$

8. $\iint_D (2x + y)\, dA$, $D = \{(x, y) \mid 1 \leqslant y \leqslant 2, y - 1 \leqslant x \leqslant 1\}$

9. $\iint_D e^{-y^2}\, dA$, $D = \{(x, y) \mid 0 \leqslant y \leqslant 3, 0 \leqslant x \leqslant y\}$

10. $\iint_D y\sqrt{x^2 - y^2}\, dA$, $D = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant x\}$

11. Draw an example of a region that is
    (a) type I but not type II
    (b) type II but not type I

12. Draw an example of a region that is
    (a) both type I and type II
    (b) neither type I nor type II

13–14 Express D as a region of type I and also as a region of type II. Then evaluate the double integral in two ways.

13. $\iint_D x\, dA$, D is enclosed by the lines $y = x$, $y = 0$, $x = 1$

14. $\iint_D xy\, dA$, D is enclosed by the curves $y = x^2$, $y = 3x$

15–16 Set up iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

15. $\iint_D y\, dA$, D is bounded by $y = x - 2$, $x = y^2$

16. $\iint_D y^2 e^{xy}\, dA$, D is bounded by $y = x$, $y = 4$, $x = 0$

17–22 Evaluate the double integral.

17. $\iint_D x \cos y\, dA$, D is bounded by $y = 0$, $y = x^2$, $x = 1$

18. $\iint_D (x^2 + 2y)\, dA$, D is bounded by $y = x$, $y = x^3$, $x \geqslant 0$

19. $\iint_D y^2\, dA$,
    D is the triangular region with vertices (0, 1), (1, 2), (4, 1)

20. $\iint_D xy\, dA$, D is enclosed by the quarter-circle $y = \sqrt{1 - x^2}$, $x \geqslant 0$, and the axes

21. $\iint_D (2x - y)\, dA$,
    D is bounded by the circle with center the origin and radius 2

22. $\iint_D y\, dA$, D is the triangular region with vertices (0, 0), (1, 1), and (4, 0)

23–32 Find the volume of the given solid.

23. Under the plane $3x + 2y - z = 0$ and above the region enclosed by the parabolas $y = x^2$ and $x = y^2$

24. Under the surface $z = 1 + x^2y^2$ and above the region enclosed by $x = y^2$ and $x = 4$

25. Under the surface $z = xy$ and above the triangle with vertices (1, 1), (4, 1), and (1, 2)

26. Enclosed by the paraboloid $z = x^2 + y^2 + 1$ and the planes $x = 0$, $y = 0$, $z = 0$, and $x + y = 2$

27. The tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$

28. Bounded by the planes $z = x$, $y = x$, $x + y = 2$, and $z = 0$

29. Enclosed by the cylinders $z = x^2$, $y = x^2$ and the planes $z = 0$, $y = 4$

30. Bounded by the cylinder $y^2 + z^2 = 4$ and the planes $x = 2y$, $x = 0$, $z = 0$ in the first octant

31. Bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y = z$, $x = 0$, $z = 0$ in the first octant

32. Bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$

33. Use a graphing calculator or computer to estimate the x-coordinates of the points of intersection of the curves $y = x^4$ and $y = 3x - x^2$. If D is the region bounded by these curves, estimate $\iint_D x\, dA$.

34. Find the approximate volume of the solid in the first octant that is bounded by the planes $y = x$, $z = 0$, and $z = x$ and the cylinder $y = \cos x$. (Use a graphing device to estimate the points of intersection.)

35–38 Find the volume of the solid by subtracting two volumes.

35. The solid enclosed by the parabolic cylinders $y = 1 - x^2$, $y = x^2 - 1$ and the planes $x + y + z = 2$, $2x + 2y - z + 10 = 0$

36. The solid enclosed by the parabolic cylinder $y = x^2$ and the planes $z = 3y$, $z = 2 + y$

37. The solid under the plane $z = 3$, above the plane $z = y$, and between the parabolic cylinders $y = x^2$ and $y = 1 - x^2$

38. The solid in the first octant under the plane $z = x + y$, above the surface $z = xy$, and enclosed by the surfaces $x = 0$, $y = 0$, and $x^2 + y^2 = 4$

39–40 Sketch the solid whose volume is given by the iterated integral.

39. $\int_0^1 \int_0^{1-x} (1 - x - y)\, dy\, dx$

40. $\int_0^1 \int_0^{1-x^2} (1 - x)\, dy\, dx$

CAS **41–44** Use a computer algebra system to find the exact volume of the solid.

41. Under the surface $z = x^3y^4 + xy^2$ and above the region bounded by the curves $y = x^3 - x$ and $y = x^2 + x$ for $x \geqslant 0$

42. Between the paraboloids $z = 2x^2 + y^2$ and $z = 8 - x^2 - 2y^2$ and inside the cylinder $x^2 + y^2 = 1$

43. Enclosed by $z = 1 - x^2 - y^2$ and $z = 0$

44. Enclosed by $z = x^2 + y^2$ and $z = 2y$

45–50 Sketch the region of integration and change the order of integration.

45. $\int_0^1 \int_0^y f(x, y)\, dx\, dy$

46. $\int_0^2 \int_{x^2}^4 f(x, y)\, dy\, dx$

47. $\int_0^{\pi/2} \int_0^{\cos x} f(x, y)\, dy\, dx$

48. $\int_{-2}^2 \int_0^{\sqrt{4-y^2}} f(x, y)\, dx\, dy$

49. $\int_1^2 \int_0^{\ln x} f(x, y)\, dy\, dx$

50. $\int_0^1 \int_{\arctan x}^{\pi/4} f(x, y)\, dy\, dx$

51–56 Evaluate the integral by reversing the order of integration.

51. $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$

52. $\int_0^1 \int_{x^2}^1 \sqrt{y} \sin y\, dy\, dx$

53. $\int_0^1 \int_{\sqrt{x}}^1 \sqrt{y^3 + 1}\, dy\, dx$

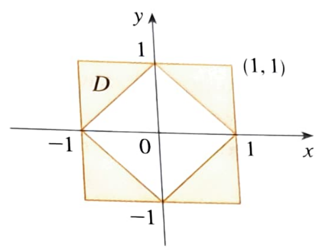
54. $\int_0^2 \int_{y/2}^1 y \cos(x^3 - 1)\, dx\, dy$

55. $\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, dx\, dy$

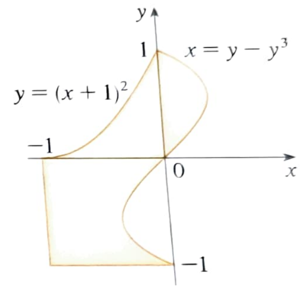
56. $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$

57–58 Express D as a union of regions of type I or type II and evaluate the integral.

57. $\iint_D x^2\, dA$



58. $\iint_D y\, dA$



59–60 Use Property 11 to estimate the value of the integral.

59. $\iint_S \sqrt{4 - x^2y^2}\, dA$, $S = \{(x, y) \mid x^2 + y^2 \leqslant 1, x \geqslant 0\}$

60. $\iint_T \sin^4(x + y)\, dA$, T is the triangle enclosed by the lines $y = 0$, $y = 2x$, and $x = 1$

61–62 Find the averge value of f over the region D.

61. $f(x, y) = xy$, D is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$

62. $f(x, y) = x \sin y$, D is enclosed by the curves $y = 0$, $y = x^2$, and $x = 1$

63. Prove Property 11.

64. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint_D f(x, y)\, dA = \int_0^1 \int_0^{2y} f(x, y)\, dx\, dy + \int_1^3 \int_0^{3-y} f(x, y)\, dx\, dy$$

Sketch the region D and express the double integral as an iterated integral with reversed order of integration.

65–69 Use geometry or symmetry, or both, to evaluate the double integral.

65. $\iint_D (x + 2)\, dA,$

$D = \{(x, y) \mid 0 \leqslant y \leqslant \sqrt{9 - x^2}\}$

66. $\iint_D \sqrt{R^2 - x^2 - y^2}\, dA,$

D is the disk with center the origin and radius R

67. $\iint_D (2x + 3y)\, dA,$

D is the rectangle $0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b$

68. $\iint_D (2 + x^2y^3 - y^2 \sin x)\, dA,$

$D = \{(x, y) \mid |x| + |y| \leqslant 1\}$

69. $\iint_D \left(ax^3 + by^3 + \sqrt{a^2 - x^2}\right) dA,$

$D = [-a, a] \times [-b, b]$

CAS **70.** Graph the solid bounded by the plane $x + y + z = 1$ and the paraboloid $z = 4 - x^2 - y^2$ and find its exact volume. (Use your CAS to do the graphing, to find the equations of the boundary curves of the region of integration, and to evaluate the double integral.)

15.3 Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\iint_R f(x, y)\, dA$, where R is one of the regions shown in Figure 1. In either case the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates.

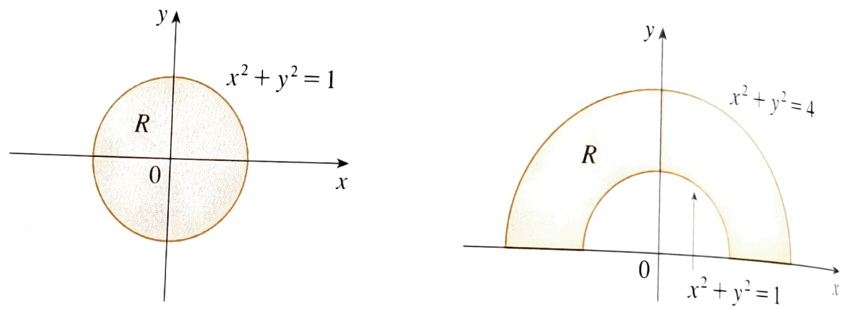
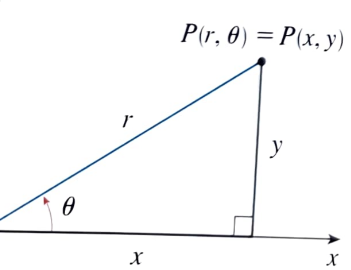


FIGURE 1

(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$

(b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$



RE 2

Recall from Figure 2 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

(See Section 10.3.)

The regions in Figure 1 are special cases of a **polar rectangle**

$$R = \{(r, \theta) \mid a \leqslant r \leqslant b, \alpha \leqslant \theta \leqslant \beta\}$$

which is shown in Figure 3. In order to compute the double integral $\iint_R f(x, y)\, dA$, where R is a polar rectangle, we divide the interval $[a, b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r = (b - a)/m$ and we divide the interval $[\alpha, \beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta = (\beta - \alpha)/n$. Then the circles $r = r_i$ and the rays $\theta = \theta_j$ divide the polar rectangle R into the small polar rectangles R_{ij} shown in Figure 4.