Math 244 Summer 2020 Midterm

August 14, 2020

- Please read each problem carefuly and follow the directions of the problem.
- You may answer all the question on this exam, or on a separate medium.
- You have 24 hours to complete this exam. It is due Sunday 8/15/2020 at 12am.
- Authorized aid: the textbook for this course, class notes for this course, and you may use a scientific calculator. However, all answer must be exact, no rounding. All other aid is unauthorized: graphing calculators, google, friends and family, SymboLab, MatLab, notes from other courses, etc.
- Show all of your work. Leaps in logic without justification will not receive credit.
- Please submit clear, organized, legible work using common file formats.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	13	
7	8	
8	8	
9	11	
Total:	90	

Name and section: ______________________________

1. (10 points) True/False. You do not have to justify your answer.

   (a) For any function $f : \mathbb{R}^3 \to \mathbb{R}$ with continuous third partial derivatives and for any surface S

   $$\iint_S \nabla \cdot (\nabla \times \nabla f) \cdot d\mathbf{S} = 0.$$

   (b) The vector field $\mathbf{F} = e^{y+2z}\mathbf{i} + xe^{y+2z}\mathbf{j} + 2xe^{y+2z}\mathbf{k}$ is conservative.

   (c) The punctured disk without boundary

   $$R = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 4\}$$

   is open, connected, and simply connected.

   (d) Suppose $f(x, y)$ is a function that satisfies

   $$\left|\frac{\partial f}{\partial x}\right| < 1 \text{ and } \left|\frac{\partial f}{\partial y}\right| < 1$$

   on the unit disk. Then we can conclude that the surface area of the graph $z = f(x, y)$ over the unit disk is at most $\sqrt{3}\pi$.

   (e) If C is any curve in $\mathbb{R}^3$ that starts on one coordinate plane and ends at the same or different coordinate plane, then

   $$\int_C \nabla(e^{xyz}) \cdot d\mathbf{r} = 0.$$

2. Let $E \subset \mathbb{R}^3$ be a simple solid and let $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ be the position vector.

(a) (2 points) Use the divergence theorem to establish the volume formula

$$\mathrm{Vol}(E) = \frac{1}{3} \iint_{\partial E} \mathbf{x} \cdot d\mathbf{S}.$$

(b) (8 points) Verify that the formula above works for the cylinder

$$E = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 \leq 4, 0 \leq z \leq 3\}.$$

3. (10 points) Let S be the elliptical shell

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + 4y^2 + 4z^2 = 4,\ z \geq 0\}$$

oriented upward. Let

$$\mathbf{F} = (y^2 - 1)\mathbf{i} + x\mathbf{j} + \tan^{-1}(xyz)(x^2 + y^2)^{3/2}\mathbf{k}.$$

Evaluate

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

4. Let $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ and $\mathbf{G} = G_1\mathbf{i} + G_2\mathbf{j} + G_3\mathbf{k}$ be two vector fields with differentiable component functions F_i and G_i. Let $c \in \mathbb{R}$ be a scalar. Then

$$\mathbf{F} + c \cdot \mathbf{G} = (F_1 + c \cdot G_2)\mathbf{i} + (F_2 + c \cdot G_2)\mathbf{j} + (F_3 + c \cdot G_3)\mathbf{k}$$

Prove the following identities

(a) (5 points) $\text{div}(\mathbf{F} + c \cdot \mathbf{G}) = \text{div}(\mathbf{F}) + c \cdot \text{div}(\mathbf{G})$.

(b) (5 points) $\text{curl}(\mathbf{F} + c \cdot \mathbf{G}) = \text{curl}(\mathbf{F}) + c \cdot \text{curl}(\mathbf{G})$.

5. (10 points) Find a potential function for the conservative vector field $\mathbf{F} = (\frac{y^2}{x} - yz)\mathbf{i} + (2y \ln x - xz)\mathbf{j} - xy\mathbf{k}$.

6. Let S be the triangular surface parametrized by

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + (2 - y - \frac{2}{3}x)\mathbf{k},$$

where (x, y) lies in the region bounded by the triangle in the xy-plane with vertices (0.0), $(3, 0)$, and $(0, 1)$.

(a) (5 points) Compute the surface area of S.

(b) (8 points) Compute the flux of the vector field $\mathbf{F} = \nabla(\ln(x+y+z))$ across S in the upward direction.

7. (8 points) Find the work done by the force field $\mathbf{F} = 2xy\mathbf{i} + y^2\mathbf{j} + \mathbf{k}$ in moving a particle along a straight line path from the point $(5, 0, 0)$ to the point $(1, 3, 2)$.

8. (8 points) Find the flux of the vector field

$$\mathbf{F} = (y^2 + z^4)\mathbf{i} - zx^3\mathbf{j} + (x^2 + y^2 + z^2)^{3/2}\mathbf{k}$$

across the boundary of the solid region given by $x^2 + y^2 + z^2 \leq 1$, $x \geq 0$, and $y \geq 0$.

9. A *unit-speed* parametrization $\mathbf{r} : [a, b] \to \mathbb{R}^3$ of a curve is one that satisfies $|\mathbf{r}'(t)| = 1$ for all $a \leq t \leq b$. One procedure for obtaining a unit-speed parametrization is as follows:

   1. Start with a parametrization $\mathbf{r} : [a, b] \to \mathbb{R}^3$ of your curve C.
   2. Compute the arc length function $s(t) = \int_a^t |\mathbf{r}'(\tau)| d\tau$.
   3. Compute the inverse function $s^{-1}(t)$.
   4. Finally, the composition $\mathbf{r}(s^{-1}(t))$ where $0 \leq t \leq s(b)$ is unit-speed.

   For a unit-speed curve C, the *curvature* of C is a function $\kappa : C \to [0, \infty)$ given by

   $$\kappa(\mathbf{r}(t)) = |\mathbf{r}''(t)|.$$

   The *total curvature* of C is defined as the line integral

   $$\int_C \kappa(x, y, z) ds.$$

   (a) (3 points) Prove step (4), that is, show that

   $$\left| \frac{d}{dt} \mathbf{r}(s^{-1}(t)) \right| = 1.$$

   Hint: chain rule, the derivative rule for inverses, and the FTC.

   (b) (4 points) Follow steps (1)-(4) to find a unit-speed parametrization for a circle of radius 3 on the xy-plane centered at the origin. Careful, the domain of the new parametrization is the range of the arc length function.

   (c) (4 points) Use the parametrization from the previous part to show that the total curvature of a circle of radius 3 is 2π. In fact, every circle has total curvature 2π.