16.4

#27. $$\vec{F} = \frac{2xy\,\vec{i} + (y^2 - x^2)\,\vec{j}}{(x^2+y^2)^2}$$

Let $$P = \frac{2xy}{(x^2+y^2)^2} \quad \text{and} \quad Q = \frac{y^2-x^2}{(x^2+y^2)^2}$$

so
$$Q_x = \frac{(x^2+y^2)^2(-2x) - (y^2-x^2)\left[2(x^2+y^2)\cdot 2x\right]}{(x^2+y^2)^4}$$

$$= \frac{(x^2+y^2)\left[(x^2+y^2)(-2x) - (y^2-x^2)4x\right]}{(x^2+y^2)^4}$$

$$= \frac{-2x^3 - 2xy^2 - 4xy^2 + 4x^3}{(x^2+y^2)^3}$$

$$= \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

$$P_y = \frac{(x^2+y^2)^2(2x) - 2xy\left(2(x^2+y^2)\,2y\right)}{(x^2+y^2)^2}$$

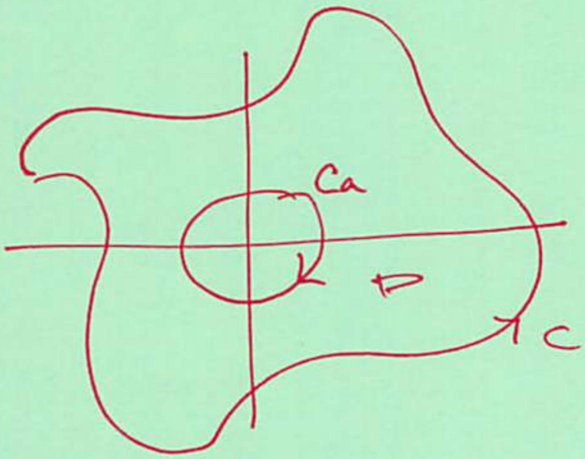
$$= \frac{(x^2+y^2)\left[(x^2+y^2)(2x) - 2xy(4y)\right]}{(x^2+y^2)^2}$$

$$= \frac{2x^3 + 2xy^2 - 8xy^2}{(x^2+y^2)^3}$$

$$= \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

$$\therefore \; Q_x - P_y = 0.$$

Let C be a positively oriented simple closed path that encloses the circle ~~of~~ C_a of radius a. oriented clockwise



Let D be the region outside of C_a and inside of C. Green's thm applies to $\vec{F}$ and D

So

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA = \iint_D 0 \, dA = 0$$

on the other hand

$$\int_{\partial D} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_{C_a} \vec{F} \cdot d\vec{r}$$

Hence,

$$\int_C \vec{F}\cdot d\vec{r} = -\int_{C_a} \vec{F}\cdot d\vec{r}$$

Parametrize $-C_a$:

$$\vec{r}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}, \qquad 0 \le t \le 2\pi$$

$$\vec{r}\,'(t) = -a\sin t\,\vec{i} + a\cos t\,\vec{j}.$$

$$\vec{F}(\vec{r}(t)) = \frac{2(a\cos t)(a\sin t)\,\vec{i} + ((a\sin t)^2 - (a\cos t)^2)\,\vec{j}}{((a\cos t)^2 + (a\sin t)^2)^2}$$

$$= \frac{2a^2\cos t\sin t\,\vec{i} + a^2(\sin^2 t - \cos^2 t)\,\vec{j}}{a^4}$$

$$= \frac{2\cos t\sin t\,\vec{i} + (\sin^2 t - \cos^2 t)\,\vec{j}}{a^2}$$

$$\vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t) = \frac{-2\cos t\sin^2 t + \cos t\sin^2 t - \cos^3 t}{a}$$

$$= \frac{-\cos t\sin^2 t - \cos^3 t}{a}$$

Thus

$$\int_{C} \vec{F} \cdot d\vec{r} = - \int_{C_a} \vec{F} \cdot d\vec{r} = \int_{-C_a} \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \frac{-\cos t \sin^2 t - \cos^3 t}{a} \, dt$$

$$= \int_0^{2\pi} \frac{-\cos t \sin^2 t}{a} \, dt - \frac{1}{a} \int_0^{2\pi} \cos t \left(1 - \sin^2 t\right) dt$$

u-sub $u = \sin t$

in both cases

$u(0) = \sin(0) = 0$

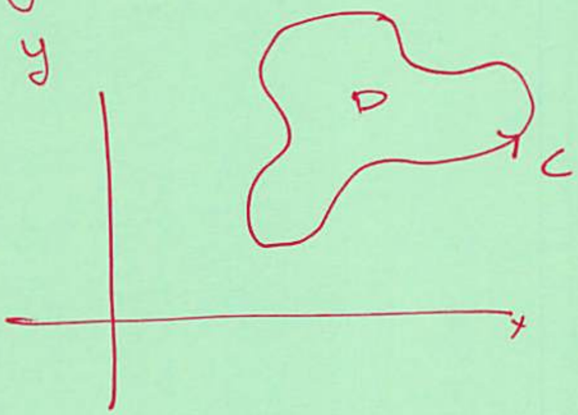
$u(2\pi) = \sin(2\pi) = 0$

Both integrals vanish

$= 0.$

#2a. $\vec{F} = \dfrac{-y\vec{i} + x\vec{j}}{x^2+y^2}$, let C

be any simple closed path that doesn't contain the origin



We showed in class $\text{curl}\,\vec{F} = \vec{0}$. Green's theorem applies to this $\vec{F}$ and D so

$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \text{curl}\,\vec{F}\cdot\vec{k}\, dA$$

$$= \iint_D \vec{0}\cdot\vec{k}\, dA$$

$$= \iint_D 0\, dA = 0.$$

16.5

#7. $\vec{F} = \langle e^x \sin y, e^y \sin z, e^z \sin x \rangle$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ D_x & D_y & D_z \\ e^x \sin y & e^y \sin z & e^z \sin x \end{vmatrix}$$

$$= \vec{i}\left(0 - e^y \cos z\right) - \vec{j}\left(e^z \cos x - 0\right)$$

$$+ \vec{k}\left(0 - e^x \cos y\right)$$

$$= -e^y \cos z\,\vec{i} - e^z \cos x\,\vec{j} - e^x \cos y\,\vec{k}$$

$$\text{div}\,\vec{F} = D_x\left(e^x \sin y\right) + D_y\left(e^y \sin z\right) + D_z\left(e^z \sin x\right)$$

$$= e^x \sin y + e^y \sin z + e^z \sin x$$

#9.

(a) div $F > 0$

(b) curl $\vec{F} = \vec{0}$

#11.

(a) div $\vec{F} = 0$

(b) curl $\vec{F} \neq 0$, curl $\vec{F}$ points into the plane.