Math 307 Exam 2, Fall 2022

Name:

Question	Points	Score
1	10	
2	9	
3	5	
4	5	
5	10	
6	13	
7	0	
Total:	52	

- You have 75 minutes to complete this exam.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work, unless otherwise indicated. You will get almost no credit for solutions that are not fully justified.
- You may not use notes or calculators on this exam.
- The back side of each page can be used as scratch work or for longer solutions. Please indicate clearly if your solution extends to the back side of a page.

1. (10 points) True/False questions. No justification necessary.

(a) True False The eigenvalues of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

are $1, 7, 2$ and 5.

(b) True False If $T : V \to W$ is a linear transformation and $c_1v_1 + \cdots + c_nv_n$ is a linear combination in V, then

$$T(c_1v_1 + \cdots + c_nv_n) = c_1T(v_1) + \cdots + c_nT(v_n).$$

(c) True False $\dim(P_9) = 9$.

(d) True False If $T, S : V \to V$ are linear operators, then $TS = ST$.

(e) True False If α and β are bases for V and P is the change of basis matrix from α to β, then

$$P[v]_\beta = [v]_\alpha,$$

for all $v \in V$.

(f) True False If A is a 11×12 matrix whose nullspace has dimension 4, then the dimension of the columnspace of A is 8.

(g) True False If y_1, y_2 are $C^\infty(\mathbb{R})$ and c is some scalar, then

$$w(cy_1, cy_2) = c^2w(y_1, y_2).$$

(h) True False The dimension of the kernel of the linear differential operator

$$(1 + x)D^3 + x^2D^4 + D^2 - 1,$$

is 3.

(i) True False The matrix of a linear transformation with respect to any pair of bases is always invertible.

(j) True False The dimension of the columnspace of

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

is 4.

2. (9 points) Multiple choice. No justification is necessary.

(a) Determine which of the following maps are linear transformations.

Mapping	Is linear	Is NOT linear
$T : \mathbb{R} \to \mathbb{R}$; $T(x) = 2x + 1$		
$T : M_{n\times n}(\mathbb{R}) \to \mathbb{R}$; $T(A) = \det(A)$		
$T : P_2 \to P_2$; $T(ax^2 + bx + c) = 2ax + b$		
$T : M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$; $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a + d$		
$T : \mathbb{R}^2 \to \mathbb{R}^2$; $T(e_1) = -e_2$ and $T(e_2) = e_1$, where e_1 and e_2 are the standard basis.		

(b) Match the n-th order linear differential operator $L : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ with a list of functions that may form a basis for $\ker(L)$.

Letter	L
	$(D^2 + 1)^2$
	$D^2 - D - 2$
	$D^2 - 4D + 5$
	$(D - 2)^3$

Letter	Basis
A	e^{-x}, e^{2x}
B	e^{2x}, xe^{2x}, x^2e^{2x}
C	$\cos x$, $\sin x$, $x\cos x$, $x\sin x$
D	$e^{2x}\cos x$, $e^{2x}\sin x$

3. (5 points) Compute a basis for the columnspace of

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

4. (5 points) Show that the functions $y_1 = e^x \cos(2x)$ and $y_2 = e^x \sin(2x)$ are linearly independent in $C^\infty(\mathbb{R})$.

5. (10 points) Let

$$A = \begin{bmatrix} 5 & 1 & -3 \\ 3 & 3 & -3 \\ 3 & 1 & -1 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Compute a basis for ONE of the eigenspaces.

6. (13 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the matrix transformation

$$T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Let α denote the standard basis of $\mathbb{R}^3$ and let β be the basis

$$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Compute the following:

(a) The change of basis matrix from α to β and its inverse.

(b) The matrix of T with respect to α.

(c) The matrix of T with respect to β.

7. (5 points (bonus)) Do not attempt this question unless you've answered every other question on this exam; I will not grade it if there is any blank problems on your exam. The *dual* of a vector space V is the vector space $V^* = L(V, \mathbb{R})$.

   (a) Show that if $T : V \to W$ is linear, then the map $T^* : W^* \to V^*$ given by

   $$T^*(\phi) = \phi T$$

   is linear.

   (b) If $\alpha = \{v_1, \ldots, v_n\}$ is a basis for show that the maps $v_j^* : V \to \mathbb{R}$ given by

   $$v_j^*(v_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

   are in V^*.

   (c) Show $\alpha^* = \{v_1^*, \ldots, v_n^*\}$ is a basis for V^*

   (d) Let $\beta = \{w_1, \ldots, w_m\}$ be a basis for W. Show that $[T^*]_{\beta^*}^{\alpha^*} = ([T]_{\alpha}^{\beta})^T$.