Math 307 Exam 2, Fall 2022

Name:

Question	Points	Score
1	10	
2	0	
3	0	
4	0	
5	0	
6	0	
7	0	
Total:	10	

- You have 75 minutes to complete this exam.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work, unless otherwise indicated. You will get almost no credit for solutions that are not fully justified.
- You may not use notes or calculators on this exam.
- The back side of each page can be used as scratch work or for longer solutions. Please indicate clearly if your solution extends to the back side of a page.

1. (10 points) True/False questions. No justification necessary.

(a) True False The dimension of $M_{3\times 4}(\mathbb{R})$ is 12.

(b) True False If A is a 33×24 matrix and the dimension of $CS(A)$ is 17, then the dimension of $NS(A)$ is 16.

(c) True False If $y_1, \ldots y_n \in C^\infty(a,b)$ and there is a point $x_0 \in (a,b)$ such that the Wronskian $w(y_1(x_0), \ldots, y_n(x_0)) \neq 0$, then $y_1, \ldots, y_n$ are linearly independent.

(d) True False A linear transformation is completely determined by its action on a basis.

(e) True False There is a linear transformation $T : V \to W$ such that

$$T(v) = 2T(v),$$

for all $v \in V$.

(f) True False The dimension of the kernel of the differential operator

$$xD^{19} + \cos x D^{13} + e^x D^7 - 102D^3 + \sin x D + 1$$

on $C(1,2)$ is 19.

(g) True False There is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(v) \neq Av$ for all $A \in M_{m \times n}(\mathbb{R})$.

(h) True False Let α and β be bases of the vector space V. If P is the change of basis matrix from α to β, then P^{-1} is the change of basis matrix from β to α.

(i) True False If v is an eigenvector of A with eigenvalue 2, then $7v$ is an eigenvector of A with eigenvalue 14.

(j) True False If 0 is an eigenvalue of the matrix A, then $\det(A) = 0$.

(k) True False If the homogeneous systen $A\vec{x} = \vec{0}$ has 3 free variables, then the dimension of the nullspace of A is 3.

(l) True False Every invertible matrix can be viewed as a change of basis matrix.

(m) True False If A is upper triangular, then the eigenvalues of A are the diagonal entries.

(n) True False If V is a finite dimensional vector space and α is a basis, then the coordinate mapping $v \mapsto [v]_\alpha$ is a linear transformation.

(o) True False If A is a 7×9 matrix, then the $NS(A)$ is a subspace of $\mathbb{R}^9$ and $CS(A)$ is a subspace of $\mathbb{R}^7$.

(p) True False There is a 7×7 matrix without any real eigenvalues.

2. Prove that the following maps are linear transformations.

1. $T: C[0,1] \to \mathbb{R}$; $T(f(x)) = \int_0^1 f(x)e^{2\pi x}dx$
2. $T: P_2 \to \mathbb{R}$; $T(p(x)) = p(9)$
3. $T: \mathbb{R}^3 \to \mathbb{R}$; $T(v) = u^T v$ where u is a fixed vector in R^3.
4. $T: V \to \mathbb{R}^n$; $T(v) = [v]_\alpha$ where α is some basis of V.

(1) $T(cf + dg) = \int_0^1 (cf+dg)(x)\, e^{2\pi x}\, dx$

$= \int_0^1 (cf(x) + dg(x))e^{2\pi x}\, dx$

$= c\int_0^1 f(x) e^{2\pi x}\, dx + d\cdot \int_0^1 g(x) e^{2\pi x}\, dx$

$= c\cdot T(f) + d \cdot T(g)$

(2) $T(p+q) = (p+q)(9) = p(9) + q(9) = T(p) + T(q)$

$T(cp) = (cp)(9) = c\cdot p(9) = c\cdot T(p)$

(3) $T(cv + dw) = u^T(cv + dw) = u^T(cv) + u^T(dw)$

$= c\cdot(u^T v) + d\cdot(u^T w)$

$= c\cdot T(v) + d\cdot T(w)$

(4) Let $\alpha = \{v_1, \dots, v_n\}$, $v = c_1v_1 + \dots + c_nv_n$ & $w = d_1v_1 + \dots + d_nv_n$

$\longrightarrow v + w = (c_1 + d_1)v_1 + \dots + (c_n + d_n)v_n$

so $[v]_\alpha + [w]_\alpha = [v+w]_\alpha$ since $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix}$

and $cv = (cc_1)v_1 + \dots + (cc_n)v_n$

so $[cv]_\alpha = c[v]_\alpha$ since $\begin{bmatrix} cc_1 \\ \vdots \\ cc_n \end{bmatrix} = c\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

3. Find bases for the kernels of the following differential operators. Justify your answer.

1. $D^2 - 10D + 24$
2. $(D-1)^3$
3. $D^2 - 4D + 5$

(1) $p(\lambda) = \lambda^2 - 10\lambda + 24 = (\lambda - 4)(\lambda - 6) = 0$

$$\Longrightarrow \lambda = 4, 6$$

Claim: $y_1 = e^{4x}$ & $y_2 = e^{6x}$ are a basis.

$$w(y_1, y_2) = \begin{vmatrix} e^{4x} & e^{6x} \\ 4e^{4x} & 6e^{6x} \end{vmatrix} = 6e^{10x} - 4e^{10x} = 2e^{10x}$$

$w(y_1, y_2) \neq 0 \Longrightarrow y_1$ & y_2 are L.I.

Since $\dim(\ker(D^2 - 10D + 24)) = 2$, then y_1 & y_2 are a basis.

(2) $p(\lambda) = (\lambda - 1)^3 = 0 \Longrightarrow \lambda = 1$ (mult. 3)

$\Longrightarrow y_1 = e^x$, $y_2 = xe^x$, & $y_3 = x^2 e^x$ are in $\ker((D-1)^3)$.

Claim: y_1, y_2, y_3 are a basis

$$w(y_1, y_2, y_3) = \begin{vmatrix} e^x & xe^x & x^2 e^x \\ e^x & e^x + xe^x & 2xe^x + x^2 e^x \\ e^x & 2e^x + xe^x & 2e^x + 4xe^x + x^2 e^x \end{vmatrix}$$

(more space)

$$= e^{3x} \begin{vmatrix} 1 & x & x^2 \\ 1 & 1+x & 2x+x^2 \\ 1 & 2+x & 2+4x+x^2 \end{vmatrix}$$

$$= e^{3x} \cdot \text{polynomial}$$

can only have finitely many zeros

$\Rightarrow y_1, y_2, y_3$ are L.I.

Since $\dim((D-1)^3) = 3$, then y_1, y_2, y_3 are a basis

(3) $p(\lambda) = \lambda^2 - 4\lambda + 5 = 0 \Rightarrow \lambda = \dfrac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 5}}{2}$

$= 2 \pm i$

choose $2+i$

$\Rightarrow y_1 = e^{2x} \cdot \cos(x)$ & $y_2 = e^{2x}\sin(x)$ are in $\operatorname{Ker}(D^2 - 4D + 5)$

<u>claim</u>: y_1, y_2 are a basis.

$$W(y_1, y_2) = \begin{vmatrix} e^{2x}\cos x & e^{2x}\sin x \\ 2e^{2x}\cos x - e^{2x}\sin x & 2e^{2x}\sin x + e^{2x}\cos x \end{vmatrix}$$

$$= e^{4x}\left(2\cos x \sin x + \cos^2 x - 2\cos x \sin x + \sin^2 x\right)$$

$$= e^{4x} \neq 0$$

$\Rightarrow y_1$ & y_2 are L.I.

since $\dim(\operatorname{Ker}(D^2 - 4D + 5)) = 2$, then

y_1 & y_2 are a basis

4. Compute bases for the nullspace, columnspace, and rowspace for the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \end{bmatrix}$$

Justify why they are bases.

$$[A \mid 0] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -4 & 0 \\ 0 & 0 & 1 & -4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\begin{array}{ccccc} D & F & D & F & F \end{array}$$

let x_2, x_4, x_5 be free

$$x_1 = -2x_2 - 3x_4 + 4x_5$$

$$x_3 = 4x_4 - 5x_5$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in NS(A) \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_4 + 4x_5 \\ x_2 \\ 4x_4 - 5x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

claim: $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $NS(A)$.

(1) the vectors span by the above equality.

(2) They are L.I. since

(more space)

$$\therefore \text{ if } \quad x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Longrightarrow$ necessarily $x_2 = x_4 = x_5 = 0.$

Claim: $\begin{bmatrix} 1 & 2 & 0 & 3 & -4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -4 & 5 \end{bmatrix}$ is a basis for $RS(A)$

Let B denote the rowspace of the row-reduced echelon form of A. The vectors span the rowspace of B as they are the only two nonzero vectors. They are L.I. since if

$$c_1 \begin{bmatrix} 1 & 2 & 0 & 3 & -4 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 0 & 1 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow c_1 = c_2 = 0$ (look at 1st & 3rd columns.)

Finally, ~~then~~ since A is row equivalent to B, then they have the same row space.

$$A^T = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -2 & 8 \\ 2 & -1 & 5 \\ -5 & 1 & -8 \\ 6 & -1 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -3 \end{bmatrix}$ is a basis of $RS(A^T)$ by similar reasoning as above

$\Longrightarrow \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ is a basis of $CS(A)$ by def.

5. Compute the Wronskian of the functions $y_1 = 9\sin(2x)$ and $y_2 = 3\cos^2(x) - 3\sin^2(x)$. Based on your solution, are the functions linearly indpenedent or is the test inconclusive?

$y_1 = 9\sin 2x$, $y_2 = 3\cos^2 x - 3\sin^2 x = 3\cos(2x)$

$$W(y_1, y_2) = \begin{vmatrix} 9\sin(2x) & 3\cos(2x) \\ 9.2\cos(2x) & -3.2\sin(2x) \end{vmatrix}$$

$$= 3^2 \begin{vmatrix} 3\sin(2x) & \cos(2x) \\ 6\cos(2x) & -2\sin(2x) \end{vmatrix}$$

$$= 9\left(-6\sin^2(2x) - 6\cos^2(2x)\right)$$

$$= -54.$$

$$\neq 0$$

$\therefore$ y_1 & y_2 are L.I

6. Find the eigenvalues and eigenspaces of the matrix

$$A = \begin{bmatrix} -3 & 2 & 1 \\ -8 & 5 & 2 \\ -4 & 2 & 2 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda+3 & -2 & -1 \\ 8 & \lambda-5 & -2 \\ 4 & -2 & \lambda-2 \end{vmatrix}$$

$$= (\lambda+3)\left[(\lambda-5)(\lambda-2) - 4\right] + 2\left[8(\lambda-2) + 8\right] - 1\left[-16 - 4(\lambda-5)\right]$$

$$= (\lambda+3)\left[\lambda^2 - 7\lambda + 6\right] + 16\left[\lambda-1\right] + 4\left[\lambda-1\right]$$

$$= (\lambda+3)(\lambda-6)(\lambda-1) + 20(\lambda-1)$$

$$= \left[(\lambda+3)(\lambda-6) + 20\right](\lambda-1)$$

$$= \left[\lambda^2 - 3\lambda + 2\right](\lambda-1)$$

$$= (\lambda-2)(\lambda-1)^2$$

$$= 0$$

$\Rightarrow \lambda = 1, 2$ (eigenvalues)

next. $\longrightarrow$

(more space)

$$\lambda = 1: \quad [I - A \mid 0] = \begin{bmatrix} 4 & -2 & -1 \\ 8 & -4 & -2 \\ 4 & -2 & -1 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array}$$

$$\longrightarrow \begin{bmatrix} 1 & -1/2 & -1/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

x_2, x_3 free & $x_1 = \tfrac{1}{2}x_2 + \tfrac{1}{4}x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in E_1 \iff \begin{bmatrix} \tfrac{1}{2}x_2 + \tfrac{1}{4}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/4 \\ 0 \\ 1 \end{bmatrix}$$

so $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/4 \\ 0 \\ 1 \end{bmatrix}$ is a basis for E_1

$$\lambda = 2: \quad [2I - A \mid 0] = \begin{bmatrix} 5 & -2 & -1 \\ 8 & -3 & -2 \\ 4 & -2 & 0 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_3 free $\quad x_1 = x_3 \quad$ & $\quad x_2 = 2x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in E_2 \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is a basis for E_2

7. Let V be the subspace of $C^\infty(\mathbb{R})$ spanned by the vectors $\cos(2x)$ and $\sin(x)$. Define the differential operator $L : V \to V$ by $L = D^2 + 4$. Let $\alpha = \{\cos(2x), \sin(x)\}$ and $\beta = \{\cos(2x) + \sin(x), \cos(2x) - \sin(x)\}$. Let $v = 2\cos(2x) - \sin(x)$.

   (a) Prove that α and β are bases for V.

   (b) Find $[v]_\alpha$.

   (c) Find $[L]_\alpha^\alpha$

   (d) Find the change of basis matrix from α to β.

   (e) Find the change of basis matrix from β to α.

   (f) Find $[L]_\beta^\beta$ using parts (c), (d), and (e).

   (g) Compute $[Lv]_\beta$.

(b) $[v]_\alpha = [2\cos(2x) - \sin(x)]_\alpha = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

(c) $[L]_\alpha^\alpha = \Big[[L\cos(2x)]_\alpha \;\; [L\sin(x)]_\alpha \Big]$

$= \Big[[-4\cos(2x) + 4\cos(2x)]_\alpha \;\; [-\sin(x) + 4\sin(x)]_\alpha \Big]$

$= \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$

(d) $P = [I]_\beta^\alpha = \Big[[\cos(2x) + \sin(x)]_\alpha \;\; [\cos(2x) - \sin(x)]_\alpha \Big]$

$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

(e) $\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] R_1 - R_2 \to \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & -1 \end{array}\right] -2R_1 + R_2$

$\to \left[\begin{array}{cc|cc} -2 & 0 & -1 & -1 \\ 0 & 2 & 1 & -1 \end{array}\right]$

(more space)

$$\Longrightarrow \quad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

(f) $$[L]_{\beta}^{\beta} = P^{-1}[L]_{\alpha}^{\alpha} P$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

$P^{-1}[Lv]_{\alpha}$

(g) $$[Lv]_{\beta} = P^{-1}[L]_{\alpha}^{\alpha}[v]_{\alpha}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -3 \\ 3 \end{bmatrix}$$