Math 307 Exam 2, Fall 2022

Name:

Question	Points	Score
1	10	
2	0	
3	0	
4	0	
5	0	
6	0	
7	0	
Total:	10	

- You have 75 minutes to complete this exam.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work, unless otherwise indicated. You will get almost no credit for solutions that are not fully justified.
- You may not use notes or calculators on this exam.
- The back side of each page can be used as scratch work or for longer solutions. Please indicate clearly if your solution extends to the back side of a page.

1. (10 points) True/False questions. No justification necessary.

(a) True False The dimension of $M_{3\times 4}(\mathbb{R})$ is 12.

(b) True False If A is a 33×24 matrix and the dimension of $CS(A)$ is 17, then the dimension of $NS(A)$ is 16.

(c) True False If $y_1, \dots y_n \in C^\infty(a, b)$ and there is a point $x_0 \in (a, b)$ such that the Wronskian $w(y_1(x_0), \dots, y_n(x_0)) \neq 0$, then $y_1, \dots, y_n$ are linearly independent.

(d) True False A linear transformation is completely determined by its action on a basis.

(e) True False There is a linear transformation $T : V \to W$ such that

$$T(v) = 2T(v),$$

for all $v \in V$.

(f) True False The dimension of the kernel of the differential operator

$$xD^{19} + \cos xD^{13} + e^x D^7 - 102D^3 + \sin xD + 1$$

on $C(1, 2)$ is 19.

(g) True False There is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that $T(v) \neq Av$ for all $A \in M_{m\times n}(\mathbb{R})$.

(h) True False Let α and β be bases of the vector space V. If P is the change of basis matrix from α to β, then P^{-1} is the change of basis matrix from β to α.

(i) True False If v is an eigenvector of A with eigenvalue 2, then $7v$ is an eigenvector of A with eigenvalue 14.

(j) True False If 0 is an eigenvalue of the matrix A, then $\det(A) = 0$.

(k) True False If the homogeneous systen $A\vec{x} = \vec{0}$ has 3 free variables, then the dimension of the nullspace of A is 3.

(l) True False Every invertible matrix can be viewed as a change of basis matrix.

(m) True False If A is upper triangular, then the eigenvalues of A are the diagonal entries.

(n) True False If V is a finite dimensional vector space and α is a basis, then the coordinate mapping $v \mapsto [v]_\alpha$ is a linear transformation.

(o) True False If A is a 7×9 matrix, then the $NS(A)$ is a subspace of $\mathbb{R}^9$ and $CS(A)$ is a subspace of $\mathbb{R}^7$.

(p) True False There is a 7×7 matrix without any real eigenvalues.

2. Prove that the following maps are linear transformations.

   1. $T : C[0,1] \to \mathbb{R}$; $T(f(x)) = \int_0^1 f(x)e^{2\pi x}dx$
   2. $T : P_2 \to \mathbb{R}$; $T(p(x)) = p(9)$
   3. $T : \mathbb{R}^3 \to \mathbb{R}$; $T(v) = u^T v$ where u is a fixed vector in R^3.
   4. $T : V \to \mathbb{R}^n$; $T(v) = [v]_\alpha$ where α is some basis of V.

3. Find bases for the kernels of the following differential operators. Justify your answer.

   1. $D^2 - 10D + 24$
   2. $(D - 1)^3$
   3. $D^2 - 4D + 5$

(more space)

4. Compute bases for the nullspace, columnspace, and rowspace for the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \end{bmatrix}$$

Justify why they are bases.

(more space)

5. Compute the Wronskian of the functions $y_1 = 9\sin(2x)$ and $y_2 = 3\cos^2(x) - 3\sin^2(x)$. Based on your solution, are the functions linearly indpenedent or is the test inconclusive?

6. Find the eigenvalues and eigenspaces of the matrix

$$A = \begin{bmatrix} -3 & 2 & 1 \\ -8 & 5 & 2 \\ -4 & 2 & 2 \end{bmatrix}$$

(more space)

7. Let V be the subspace of $C^{\infty}(\mathbb{R})$ spanned by the vectors $\cos(2x)$ and $\sin(x)$. Define the differential operator $L : V \to V$ by $L = D^2 + 4$. Let $\alpha = \{\cos(2x), \sin(x)\}$ and $\beta = \{\cos(2x) + \sin(x), \cos(2x) - \sin(x)\}$. Let $v = 2\cos(2x) - \sin(x)$.

   (a) Prove that α and β are bases for V.

   (b) Find $[v]_{\alpha}$.

   (c) Find $[L]_{\alpha}^{\alpha}$

   (d) Find the change of basis matrix from α to β.

   (e) Find the change of basis matrix from β to α.

   (f) Find $[L]_{\beta}^{\beta}$ using parts (c), (d), and (e).

   (g) Compute $[Lv]_{\beta}$.

(more space)