1. (10 points) True/False questions. No justification necessary.

(a) True **False** There is a 5×6 matrix that row-equivalent to a matrix with 6 leading 1's.

(b) True **False** A system with 3 equations in 2 variables is always inconsistent.

(c) True **False** If A is a 3×2 matrix and B is a 3×2 matrix, then AB is a 3×2 matrix.

(d) **True** False If A is a square matrix, then $(2A)^4 = 16A^4$.

(e) **True** False If A is an invertible matrix, then the homogeneous system $A\vec{x} = \vec{0}$ only has the trivial solution.

(f) **True** False The matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

is invertible.

(g) **True** False The transpose of a 7×18 matrix is a matrix of size 18×7.

(h) **True** False For all $n \times n$ invertible matrices A and B, $(AB)^{-1} = B^{-1}A^{-1}$.

(i) True **False** If every entry of a 17×17 matrix is a 2, then the determinant of this matrix is 2^{17}.

(j) **True** False $\det(AA^T) = \det(A^2)$.

(k) **True** False The determinant of a triangular matrix is the product of the diagonal entries.

(l) True **False** For all square matrices A and B, $\det(A + B) = \det(A) + \det(B)$.

(m) **True** False There is a vector space V where $3 \cdot v = 2 \cdot v$ for all $v \in V$. $V = \{\vec{0}\}$

(n) True **False** The vector

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

is in the span of the vectors

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix},$$ ← all the ~~#~~ third entries are zero.

(o) **True** False Every vector space V has at least two subspaces, the zero vector space $\{\vec{0}\}$ and the space V itself.

(p) **True** False Every invertible matrix is a product of elementary matrices.

(q) True False To add 2 times row 3 onto row 1 of the 3×3 matrix A we multiply on the right by the matrix

Left

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(r) True False Every matrix in row-reduced echelon form has at least one 1 in one of its columns. the zero matrix

(s) True False Linear algebra is awesome!

2. The coefficient matrix of each of the following augmented matrices is in row-reduced echelon form. In the space provided write the solution set (if it exists) to the corresponding system of linear equations

Matrix	Solution Set
$\left[\begin{array}{ccccc\|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$	
$\left[\begin{array}{ccccc\|c} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$	
$\left[\begin{array}{ccccc\|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$	

3. Prove or disprove that the following sets are subspaces of $M_3(\mathbb{R})$.
   (a) The set of all upper triangular 3×3 matrices.
   (b) The set of 3×3 matrices that have a 1 in the 1,1 entry.
   (c) The set of 3×3 invertible matrices.

4. Show that the matrix

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

is invertible three different ways.

#2

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

System has no solution.

$$\begin{array}{c} \left[\begin{array}{ccccc|c} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} f & f & f & f & d \end{array} \end{array}$$

x_1, x_2, x_3, x_4 are free and $x_5 = 0$.

$$\begin{array}{c} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \\ \begin{array}{ccccc} d & d & d & d & f \end{array} \end{array}$$

x_5 is free

$x_1 = 1 - x_5$

$x_2 = 1 - x_5$

$x_3 = 1 - x_5$

$x_4 = 1 - x_5$

#3.

(a) $W = \left\{ \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix} ; \ a, b, c, d, e, f \in \mathbb{R} \right\}$

(i) Let $A, B \in W$, then

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & e_1 \\ 0 & 0 & f_1 \end{bmatrix} \text{ and } B = \begin{bmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & e_2 \\ 0 & 0 & f_2 \end{bmatrix}.$$

So

$$A + B = \begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & e_1 \\ 0 & 0 & f_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 & c_2 \\ 0 & d_2 & e_2 \\ 0 & 0 & f_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 & c_1 + c_2 \\ 0 & d_1 + d_2 & e_1 + e_2 \\ 0 & 0 & f_1 + f_2 \end{bmatrix}$$

$\in W.$

$\therefore W$ is closed under addition

(ii) Let $A \in W$ (as above) and let $c \in \mathbb{R}$, then

$$cA = c\begin{bmatrix} a_1 & b_1 & c_1 \\ 0 & d_1 & e_1 \\ 0 & 0 & f_1 \end{bmatrix} = \begin{bmatrix} ca_1 & cb_1 & cc_1 \\ 0 & cd_1 & ce_1 \\ 0 & 0 & cf_1 \end{bmatrix} \in W$$

$\therefore$ W is closed under scalar multiplication.

$\therefore$ W is a subspace.

(b) $W = \left\{ \begin{bmatrix} 1 & a & b \\ c & d & e \\ f & g & h \end{bmatrix} : a, b, c, d, e, f, g, h \in \mathbb{R} \right\}$

The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is in W, but

$$2 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ is not

$\therefore$ W is not closed under scalar multiplication.

(c) The set $W = \{A : A \text{ is invertible}\}$

The matrices $A = I_3$ and $B = -I_3$ are in W, ~~both~~ (they have nonzero determinants).

But the sum

$$A + B = I_3 + (-I_3) = O,$$

the zero matrix, is not invertible so $A+B$ is not in W.

$\therefore$ W is not closed under addition

#4 $\quad A = \begin{array}{c} \begin{array}{ccc} & & b \end{array} \\ \begin{bmatrix} 5 & 2 & 1 \\ 4 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \end{array}$

(1) $\det A = 1 \cdot \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 5 & 2 \\ 4 & 2 \end{vmatrix}$

$= 0 - 1 + 2$

$= 1$

$\neq 0 \quad \therefore$ A is invertible

(2) Solve the system $A\vec{x} = \vec{0}$:

$$\left[\begin{array}{ccc|c} 5 & 2 & 1 & 0 \\ 4 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Since the system $A\vec{x} = \vec{0}$ only has the trivial solution $\Rightarrow$ A is invertible

(3) From above $\operatorname{rref}(A) = I_3$

$\Rightarrow$ A is invertible.

(4) $\left[\begin{array}{ccc|ccc} 5 & 2 & 1 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\xrightarrow[\text{reduce}]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -1 \\ 0 & 0 & 1 & 0 & -1 & 2 \end{array}\right]$$

this is A^{-1}.

(5) Show that for each $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ the system

$A\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has a unique solution.

$$\left[\begin{array}{ccc|c} 5 & 2 & 1 & a \\ 4 & 2 & 1 & b \\ 2 & 1 & 1 & c \end{array}\right] \xrightarrow[\text{reduce}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-b \\ 0 & 1 & 0 & -2a+3b-c \\ 0 & 0 & 1 & -b+2c \end{array}\right]$$

$\therefore$ the system $A\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has the unique

solution $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a-b \\ -2a+3b-c \\ -\cancel{4}b+2c \end{bmatrix}$

$\Longleftrightarrow$ A is invertible.

#5.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 22 & 28 \\ 49 & 64 \end{bmatrix}$

(b) not defined

(c) not defined

(d) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T$

$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

$= \begin{bmatrix} 2 & 5 & 8 \\ 6 & 9 & 12 \end{bmatrix}$

(e) $$\left(\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}\right)^T + 2\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}$$

$$= \begin{bmatrix}9 & 12 & 15\\4 & 5 & 6\end{bmatrix}^T + \begin{bmatrix}2 & 4\\6 & 8\\10 & 12\end{bmatrix}$$

$$= \begin{bmatrix}9 & 4\\12 & 5\\15 & 6\end{bmatrix} + \begin{bmatrix}2 & 4\\6 & 8\\10 & 12\end{bmatrix}$$

$$= \begin{bmatrix}11 & 8\\18 & 13\\25 & 18\end{bmatrix}$$

#6.

(a) $\det A = 1 \cdot 2 \cdot 3 \cdot 1 = 6$

$$\det B = \begin{vmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 4 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

$$= -1 \cdot \; 1 \cdot 2 \cdot 3 \cdot 4$$

$$= -24$$

$$\det C = \begin{vmatrix} 2 & 3 & 0 & 2 \\ 4 & 3 & 2 & 1 \\ 6 & 0 & 0 & 3 \\ 7 & 0 & 0 & 4 \end{vmatrix}$$

$$= -2 \cdot \begin{vmatrix} 2 & 3 & 2 \\ \cancel{4} & \cancel{3} & \cancel{1} \\ 6 & 0 & 3 \\ 7 & 0 & 4 \end{vmatrix}$$

$$= (-2) \cdot (-3) \cdot \begin{vmatrix} 6 & 3 \\ 7 & 4 \end{vmatrix}$$

$$= 6 \cdot (24 - 21)$$

$$= 18$$

(b)

$$\det(ABABAB) = (\det A)^3 \cdot (\det B)^3$$

$$= (6)^3 \cdot (-24)^3$$

$$= -2985984$$

$$\det(A^T B^{-1} C^2)$$

$$= \det A \cdot \frac{1}{\det B} \cdot (\det C)^2$$

$$= 6 \cdot \frac{1}{-24} \cdot (18)^2$$

$$= -81$$

#7. Suppose

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = v$$

$$\iff (c_1 v_1 + c_2 v_2 + c_3 v_3)(x) = v(x)$$

$$\iff c_1 \cdot v_1(x) + c_2 \cdot v_2(x) + c_3 \cdot v_3(x) = v(x)$$

$$\iff c_1(x^2+x) + c_2(x+1) + c_3(x^2+1) = x^2+3x-1$$

$$\iff (c_1+c_3)x^2 + (c_1+c_2)x + (c_2+c_3) = x^2+3x-1$$

$$\iff \begin{aligned} c_1 + c_3 &= 1 \\ c_1 + c_2 &= 3 \\ c_2 + c_3 &= -1 \end{aligned} \quad \text{and} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 3 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -3/2 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/2 \\ -3/2 \end{bmatrix}$$

$\therefore$ system has a solution, so v is in the span.

#8. $x_1 + 7x_3 + 2x_6 = 0$

$3x_3 + x_6 = 0$

$$\rightsquigarrow \left[\begin{array}{cccccc|c} 1 & 0 & 7 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 - \frac{7}{3}R_2 \\ \cancel{R_2/3} \end{array}$$

$2 - \frac{7}{3} = \frac{6-7}{3} = -\frac{1}{3}$

$$\to \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} \\ R_2/3 \end{array}$$

$$\to \begin{array}{c} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & -1/3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/3 & 0 \end{array}\right] \\ \begin{array}{cccccc} d & f & d & f & f & f \end{array} \end{array}$$

Let x_2, x_4, x_5, x_6 be free, and

$x_1 = 1/3 x_6$

$x_3 = -1/3 x_6$

#9.

$$(2f+3g)(-1)$$

$$= (2f)(-1) + (3g)(-1)$$

$$= 2 \cdot f(-1) + 3 \cdot g(-1)$$

$$= 2 \cdot |-1| + 3 \cdot \left((-1)^2 - 1 \right)$$

$$= 2 + 3 \cdot 0$$

$$= 2.$$