1. True/False questions. No justification necessary.

(a) True	False	If the vectors v_1, v_2, v_3, v_4 are linearly independent in $\mathbb{R}^4$, then they must form a basis of $\mathbb{R}^4$.
(b) True	False	If $v_1 + 2v_2 + 3v_3 = 3v_1 + 2v_2 + 1v_3$, then the vectors v_1, v_2, v_3 are linearly dependent.
(c) True	False	If $w_1, \ldots, w_n$ span $\mathbb{R}^5$, then n must be 5.
(d) True	False	$\dim(P_{10}) = 10$.
(e) True	False	The dimension of the trivial vector space $\{0\}$ is 1.
(f) True	False	If $A \in M_{5\times 6}(\mathbb{R})$ has rank 2, then the dimension of the nullspace of A is 4.
(g) True	False	If A is a 3×3 invertible matrix, then the columnspace of A is $\mathbb{R}^3$.
(h) True	False	$w(e^x, xe^x, x^2e^x) = e^x w(1, x, x^2)$.
(i) True	False	If the wronskian of the functions y_1, y_2, y_3 is a constant function, then the vectors y_1, y_2, y_3 are linearly independent.
(j) True	False	If $T : V \to V$ is a linear operator, then the set of fixed points $\{v \in V : T(v) = v\}$ is a subspace of V.
(k) True	False	The map $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $T(A) = AB - BA$, where B is a fixed matrix, is a linear transformation.
(l) True	False	If T is a linear transformation and $v = c_1v_1 + c_2v_2$, then $Tv = c_1Tv_1 + c_2Tv_2$.
(m) True	False	The kernel of a linear transformation is a subspace of the codomain of the transformation.
(n) True	False	The determinant $\det : M_4(\mathbb{R}) \to \mathbb{R}$ is a linear transformation.
(o) True	False	Let $L : C^\infty(0,1) \to C^\infty(0,1)$ be the operator $$(D+x)^6 = D^{(6)}+6xD^{(5)}+15x^2D^{(4)}+20x^3D^{(3)}+15x^4D''+6x^5D'+x^6.$$ Then $\mathrm{nullity}(L) = 7$.
(p) True	False	If the characteristic polynomial of the n-th order homogeneous constant coefficient linear differential equation has the form $p(\lambda) = q(\lambda)(2\lambda+3)^4$, then $$e^{-3/2x}, xe^{-3/2x}, x^2e^{-3/2x}, x^3e^{-3/2x},$$ are solutions.
(q) True	False	$\mathrm{Re}(2e^{i\pi/6}) = 1$.
(r) True	False	Viewing $\mathbb{C}$ as a real vector space, the imaginary part map $\mathrm{Im} : \mathbb{C} \to \mathbb{R}$ is a linear transformation.
(s) True	False	Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation.

#2

characteristic polynomial: $p(\lambda) = \lambda^3 - 3\lambda^2 + \lambda - 3$

3 is a root $\Longrightarrow$ $\lambda - 3$ divides $p(\lambda)$

$$\begin{array}{r|l} & \lambda^2 + 1 \\ \hline \lambda - 3 & \lambda^3 - 3\lambda^2 + \lambda - 3 \\ & -(\lambda^3 - 3\lambda^2) \\ \hline & \qquad\qquad \lambda - 3 \\ & \qquad\qquad \lambda - 3 \\ \hline & \qquad\qquad\quad 0 \end{array}$$

so $p(\lambda) = (\lambda - 3)(\lambda^2 + 1) = (\lambda - 3)(\lambda + i)(\lambda - i)$

roots: $3, \pm i$

pick just i.

$\lambda = 3$ gives the solution $y_1 = e^{3x}$

$\lambda = i = 0 + 1 \cdot i$ gives the solutions

$y_2 = e^{0 \cdot x} \cos(1 \cdot x) = \cos x$

$y_3 = e^{0 \cdot x} \sin(1 \cdot x) = \sin x$.

Claim: y_1, y_2, y_3 are L.I.

$$W(y_1, y_2, y_2) = \begin{vmatrix} e^{3x} & \cos x & \sin x \\ 3e^{3x} & -\sin x & \cos x \\ 9e^{3x} & -\cos x & -\sin x \end{vmatrix}$$

$$= e^{3x}\left(\sin^2 x + \cos^2 x\right) - 3e^{3x}\left(-\sin x \cos x + \sin x \cos x\right) + 9e^{3x}\left(\cos^2 x + \sin^2 x\right)$$

$$= 10e^{3x}$$

$$\neq 0$$

$\therefore y_1, y_2, y_3$ are L.I.

Since the dimension of the space of solutions of $y^{(3)} - 3y'' + y' - 3y = 0$ is 3, and y_1, y_2, y_2 are L.I., then y_1, y_2, y_3 are a basis.

Hence, the general solution to the differential equation is

$$y_H = C_1 e^{3x} + C_2 \cos x + C_3 \sin x.$$

#3

$W(1, 1-x, (1-x)^2)$

$$= \begin{vmatrix} 1 & 1-x & (1-x)^2 \\ 0 & -1 & 2(1-x)(-1) \\ 0 & 0 & -2\cdot(-1) \end{vmatrix}$$

$= -2$

#4. The vectors

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}$$

are L.I. since

$$\left[\begin{array}{ccc|c}1 & 1 & 1 & 0\\0 & 2 & 2 & 0\\0 & 0 & 3 & 0\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{array}\right]$$

has rank 3

Since $\dim(\mathbb{R}^3) = 3$, then these vectors

form a basis.

#5.

$$[A \mid \vec{0}] = \left[\begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 & 0 \\ -1 & 1 & 0 & 2 & \rho & 0 \\ 1 & -1 & -2 & 0 & 3 & 0 \\ 2 & -2 & -1 & 3 & 4 & 0 \end{array}\right]$$

$$\longrightarrow \begin{array}{c} \left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \\ \begin{array}{ccccc} d & f & d & d & d \end{array} \end{array}$$

let x_2 be free, so

$x_1 = x_2$

$x_3 = 0$

$x_4 = 0$

$x_5 = 0$

Hence,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \operatorname{Ker}(A) \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is a basis for $\operatorname{Ker}(A)$.

Since $\operatorname{rref}(A) = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

then the row vectors

$\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$,

$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

are a basis for $RS(A)$

$$A^T = \begin{bmatrix} 1 & -1 & 1 & 2 \\ -1 & 1 & -1 & -2 \\ -1 & 0 & -2 & -1 \\ 1 & 2 & 0 & 3 \\ 1 & 0 & 3 & 4 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So the column vectors

$$\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for $CS(A)$.

#6.

(a) $T: V \to V$, $T\{a_1, a_2, a_3, \ldots\} = \{0, a_1, a_2, a_3, \ldots\}$

This map is a linear transformation

$T(\{a_1, a_2, \ldots\} + \{b_1, b_2, \ldots\})$

$= T(\{a_1+b_1, a_2+b_2, a_3+b_3, \ldots\})$

$= \{0, a_1+b_1, a_2+b_2, \ldots\}$

$= \{0, a_1, a_2, \ldots\} + \{0, b_1, b_2, \ldots\}$

$= T\{a_1, a_2, \ldots\} + T\{b_1, b_2, \ldots\}$

$\Longrightarrow T$ is linear

$$T(c\{a_1, a_2, \ldots\})$$

$$= T\{ca_1, ca_2, \ldots\}$$

$$= \{0, ca_1, ca_2, \ldots\}$$

$$= c \cdot \{0, a_1, a_2, \ldots\}$$

$$= c \cdot T\{a_1, a_2, \ldots\}$$

$\Rightarrow$ T is homogeneous.

$\therefore$ the map is linear

(b) $T: V \to V$, $T\{a_1, a_2, a_3, \dots\} = \{1, a_1, a_2, \dots\}$

this is not a linear transformation.

Let $c = 2$ and $\{a_1, a_2, \dots\} = \{1, 1, \dots\}$

$$T(c\{a_1, a_2, \dots\}) = T(2 \cdot \{1, 1, \dots\})$$

$$= T(\{2, 2, \dots\})$$

$$= \{1, 2, 2, \dots\}$$

on the other hand,

$$c \cdot T\{a_1, a_2, \dots\} = 2 \cdot T\{1, 1, \dots\}$$

$$= 2 \cdot \{1, 1, \dots\}$$

$$= \{2, 2, \dots\}$$

So $c \cdot T\{a_1, a_2, \dots\} \neq T(c\{a_1, a_2, \dots\})$

(c) $T: \mathcal{F}(\mathbb{R}) \to V$, $T(f(x)) = \{f(1), f(2), \dots\}$

this map is a linear transformation.

Linearity: Let $f, g \in \mathcal{F}(\mathbb{R})$, then

$$T(f+g) = \{(f+g)(1), (f+g)(2), \dots\}$$

$$= \{f(1)+g(1), f(2)+g(2), \dots\}$$

$$= \{f(1), f(2), \dots\} + \{g(1), g(2), \dots\}$$

$$= T(f) + T(g)$$

$\longrightarrow$

Homogeneity: Let $f \in \mathcal{F}(\mathbb{R})$ and $c \in \mathbb{R}$, then

$$T(cf) = \{(cf)(1), (cf)(2), \ldots\}$$

$$= \{c \cdot f(1), c \cdot f(2), \ldots\}$$

$$= c \cdot \{f(1), f(2), \ldots\}$$

$$= c \cdot T(f).$$

$\therefore$ the map is a linear transformation.

#7.

(a) Since $W(1, 1-x, (1-x)^2) = -2 \neq 0$, then

$1, 1-x, (1-x)^2$ are L.I.. Since $\dim(P_2) = 3$,

then these functions form a basis.

Let $\alpha = \{1, x, x^2\}$ and $\beta = \{1, 1-x, (1-x)^2\}$.

(b) $P = [I]_\beta^\alpha$

$$= \Big[[I(1)]_\alpha \ [I(1-x)]_\alpha \ [I((1-x)^2)]_\alpha \Big]$$

$$= \Big[[1]_\alpha \ [1-x]_\alpha \ [1-2x+x^2]_\alpha \Big]$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) $[T]_\alpha^\alpha$. $\quad T(f(x)) = f(3x-1)$

$$= \Big[[T(1)]_\alpha \ [T(x)]_\alpha \ [T(x^2)]_\alpha \Big]$$

$$= \Big[[1]_\alpha \ [3x-1]_\alpha \ [(3x-1)^2]_\alpha \Big]$$

$$= \Big[[1]_\alpha \ [3x-1]_\alpha \ [9x^2-6x+1]_\alpha \Big]$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -6 \\ 0 & 0 & 9 \end{bmatrix}$$

(d) $[T]_\beta^\beta = P^{-1} [T]_\alpha^\alpha P$

$$\left[\begin{array}{ccc|ccc} \multicolumn{3}{c|}{\overbrace{\begin{matrix}1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1\end{matrix}}^{P}} & \begin{matrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{matrix} \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} \begin{matrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{matrix} & \underbrace{\begin{matrix}1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1\end{matrix}}_{P^{-1}} \end{array}\right]$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -6 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\vdots$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -6 \\ 0 & 0 & 9 \end{bmatrix}$$

$$[4x+1]_\beta = [I(4x+1)]_\beta$$

$$= [I]_\alpha^\beta [4x+1]_\alpha$$

$$= P^{-1} \cdot \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$$