Math 307 Exam 2, Spring 2023

Name:

Question	Points	Score
1	0	
2	0	
3	0	
4	0	
5	0	
6	0	
7	0	
Total:	0	

- You have 75 minutes to complete this exam.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work, unless otherwise indicated. You will get almost no credit for solutions that are not fully justified.
- No electronic devices are authorized with the exception of a scientific calculator.
- The back side of each page can be used as scratch work or for longer solutions. Please indicate clearly if your solution extends to the back side of a page.

1. True/False questions. No justification necessary.

(a) True False If the vectors v_1, v_2, v_3, v_4 are linearly independent in $\mathbb{R}^4$, then they must form a basis of $\mathbb{R}^4$.

(b) True False If $v_1 + 2v_2 + 3v_3 = 3v_1 + 2v_2 + 1v_3$, then the vectors v_1, v_2, v_3 are linearly dependent.

(c) True False If $w_1, \ldots, w_n$ span $\mathbb{R}^5$, then n must be 5.

(d) True False $\dim(P_{10}) = 10$.

(e) True False The dimension of the trivial vector space $\{0\}$ is 1.

(f) True False If $A \in M_{5\times6}(\mathbb{R})$ has rank 2, then the dimension of the nullspace of A is 4.

(g) True False If A is a 3×3 invertible matrix, then the columnspace of A is $\mathbb{R}^3$.

(h) True False $w(e^x, xe^x, x^2e^x) = e^x w(1, x, x^2)$.

(i) True False If the wronskian of the functions y_1, y_2, y_3 is a constant function, then the vectors y_1, y_2, y_3 are linearly independent.

(j) True False If $T : V \to V$ is a linear operator, then the set of fixed points $\{v \in V : T(v) = v\}$ is a subspace of V.

(k) True False The map $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $T(A) = AB - BA$, where B is a fixed matrix, is a linear transformation.

(l) True False If T is a linear transformation and $v = c_1v_1 + c_2v_2$, then $Tv = c_1Tv_1 + c_2Tv_2$.

(m) True False The kernel of a linear transformation is a subspace of the codomain of the transformation.

(n) True False The determinant $\det : M_4(\mathbb{R}) \to \mathbb{R}$ is a linear transformation.

(o) True False Let $L : C^\infty(0, 1) \to C^\infty(0, 1)$ be the operator

$$(D+x)^6 = D^{(6)}+6xD^{(5)}+15x^2D^{(4)}+20x^3D^{(3)}+15x^4D^{''}+6x^5D^{'}+x^6.$$

Then $\text{nullity}(L) = 7$.

(p) True False If the characteristic polynomial of the n-th order homogeneous constant coefficient linear differential equation has the form $p(\lambda) = q(\lambda)(2\lambda + 3)^4$, then

$$e^{-3/2x}, xe^{-3/2x}, x^2e^{-3/2x}, x^3e^{-3/2x},$$

are solutions.

(q) True False $\text{Re}(2e^{i\pi/6}) = 1$.

(r) True False Viewing $\mathbb{C}$ as a real vector space, the imaginary part map $\text{Im} : \mathbb{C} \to \mathbb{R}$ is a linear transformation.

(s) True False Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation.

2. Find the general solution to the 3rd order constant coefficient homogeneous linear differential equation

$$y^{(3)} - 3y'' + y' - 3y = 0.$$

Hint: 3 is a root of the characteristic polynomial.

3. Compute the Wronskian of of the functions $y_1 = 1$, $y_2 = 1 - x$, and $y_3 = (1 - x)^2$.

4. From the list of vectors below find a subset of vectors that form a basis for $\mathbb{R}^3$. Justify your answer.

$$\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\3\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}0\\0\\3\end{bmatrix}.$$

5. Find basis for the nullspace, rowspace, columnspace for the matrix

$$A = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 0 & 2 & 0 \\ 1 & -1 & -2 & 0 & 3 \\ 2 & -2 & -1 & 3 & 4 \end{bmatrix}$$

6. Let V denote the vector space of all infinite sequences of real numbers $\{a_1, a_2, a_3, \ldots\}$ with operations

$$\{a_1, a_2, a_3, \ldots\} + \{b_1, b_2, b_3, \ldots\} = \{a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots\},$$

and

$$c\{a_1, a_2, a_3, \ldots\} = \{ca_1, ca_2, ca_3, \ldots\}.$$

Prove or disprove that the following maps are linear transformations.

(a) $T : V \to V$ defined by $T\{a_1, a_2, a_3, \ldots\} = \{0, a_1, a_2, a_3, \ldots\}$.

(b) $T : V \to V$ defined by $T\{a_1, a_2, a_3, \ldots\} = \{1, a_1, a_2, a_3, \ldots\}$.

(c) $T : F(\mathbb{R}) \to V$ defined by $T(f(x)) = \{f(1), f(2), f(3), \ldots\}$.

7. Let α be the standard basis of P_2. Let $\beta = \{1, 1 - x, (1 - x)^2\}$. Let T be the operator on P_2 defined by $T(f(x)) = f(3t - 1)$.

(a) Check that β is a basis of P_2.

(b) Compute the change of basis matrix form α to β.

(c) Find the matrix of T with respect to α.

(d) Find the matrix of T with respect to β.

(e) Find the coordinate vector of $[4x + 1]$ with respect to β.