1. True/False questions. No justification necessary.

(a) True False If A is a 3×5 matrix, then there is always a nonzero vector $x \in \mathbb{R}^5$ such that $Ax = 0$.

(b) True False There is an 2×2 upper triangular matrix A such that

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(c) True False Every invertible matrix is a product of elementary matrices.

(d) True False If $\det(A) = \det(A^T)$, then A must be symmetric.

(e) True False There is a real vector space with exactly 7 vectors.

(f) True False If U and W are subpaces of a vector space V, then the space of all sums

$$U + W := \{u + w : u \in U \text{ and } w \in W\}$$

is a subspace of V.

(g) True False The columns of any 3×4 matrix must be linearly dependent.

(h) True False Every basis of $M_{3\times 4}(\mathbb{R})$ has 12 matrices.

(i) True False If v_1, v_2, v_3 is a basis for V, then

$$v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_3,$$

is also a basis for V.

(j) ~~True~~ False If $W(y_1, y_2, y_3) = 0$ on (a, b), then y_1, y_2, y_3 are linearly dependent in $F(a, b)$.

(k) True False Every polynomial is a linear transformation from $\mathbb{R}$ to $\mathbb{R}$.

(l) True False If q is any polynomial, then

$$e^{-2x}, xe^{-2x}, x^2e^{-2x}, x^3e^{-2x}$$

are all in the kernel of the differential operator $q(D)(D + 2)^4$.

(m) True False Every linear transformation from $T : R^n \to \mathbb{R}^m$ is a matrix transformation $Tx = Ax$ for some matrix $A \in M_{m\times n}(\mathbb{R})$.

(n) True False If v is an eigenvector with eigenvalue 2, then $7v$ is an eigenvector with eigenvalue 14.

(o) True False If A and B are similar, then $\det(A) = \det(B)$.

(p) True False If A is diagonalizable, then $\det(A)$ is equal to the product of its eigenvalues.

(q) True False The matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

is in Jordan canonical form.

(r) True False Every matrix is similar to an upper triangular matrix.

(s) True False The dimension of the space of solutions for a homogeneous system of n first-order linear differential equations is n.

2. Solve the system of linear equations

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 - x_4 &= 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 &= 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 &= -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 &= 11 \end{aligned}$$

3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Find $(AB)^T$, BA, $(BC)^{-1}$, CB, A^{-1}, B^{-1}, $B^T + 2C$. or state that the expression is undefined.

4. Let

$$A = \begin{bmatrix} -2 & 1 & 5 & 2 \\ -3 & -1 & 0 & -1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Find $\det(2A)$ and $\det(A^{-1}B^2C^T)$.

5. Prove or disprove the following.

   (a) The set of solutions of the differential equation $y'' + \cos(x)y' - y = 0$ is a subspace of $F(\mathbb{R})$.

   (b) The set of vectors satisfying $Av = \lambda v$ for a fixed $n \times n$ matrix A is a subspace of $\mathbb{R}^n$.

   (c) The set of invertible 3×3 matrices is a subspace of $M_3(\mathbb{R})$.

6. Consider the vectors

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \right\}$$

   (a) Are the vectors in α linearly independent set of vectors? Clearly justify.

   (b) Do the vectors in α span $\mathbb{R}^3$? Clearly justify.

   (c) Does α form a basis of $\mathbb{R}^3$?

#2.

$$\left[\begin{array}{cccc|c} 4 & 3 & 2 & -1 & 4 \\ 5 & 4 & 3 & -1 & 4 \\ -2 & -2 & -1 & 2 & -3 \\ 11 & 6 & 4 & 1 & 11 \end{array}\right]$$

$$\xrightarrow{\text{RREF}} \begin{array}{c} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccccc} d & d & d & f & \end{array} \end{array}$$

let x_4 free

$$x_1 = 1 - x_4$$

$$x_2 = 2 + 3x_4$$

$$x_3 = -3 - 2x_4$$

#3

- $(AB)^T = B^T A^T$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 5 & 4 \\ 3 & 6 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 8 \\ 4 & 10 & 13 \end{bmatrix}$$

- BA not defined

- $$CB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & 10 \end{bmatrix}$$

- A^{-1} is not defined raw 1 & 3 are scalar multiples

  $\Rightarrow$ determinant is 0

  $\Rightarrow$ not invertible

- B^{-1} not defined, B is not square

- $B^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $2C = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$

$$\Rightarrow B^T + 2C = \begin{bmatrix} 2 & 5 & 6 \\ 9 & 10 & 13 \end{bmatrix}$$

#4.

$$\det A = \begin{bmatrix} -2 & 1 & 5 & 2 \\ -3 & -1 & 0 & -1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= 2 \cdot \begin{vmatrix} -2 & 1 & 2 \\ -3 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} - 1 \cdot \begin{vmatrix} 2 & 1 & 5 \\ -3 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2 \cdot \left[-2(-1+1) - 1(-3+1) + 2(-3+1) \right]$$

$$- 1 \cdot \left[5(-3+1) + 1(-2+3) \right]$$

$$= 2 \cdot \left[0 + 2 - 4 \right] - 1 \cdot \left[-10 + 1 \right]$$

$$= 2 \left[-2 \right] - 1 \left[-9 \right]$$

$$= -4 + 9$$

$$= 5$$

$$\det(B) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

upper triangular
prod. of diagonal.

$$= 1\cdot 1\cdot 1\cdot 1$$

$$= 1.$$

$$\det(C) = \begin{vmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{vmatrix} \quad R_1 \leftrightarrow R_2$$

$$= -1\cdot\begin{vmatrix} 0 & 0 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{vmatrix} \quad R_3 \leftrightarrow R_1$$

$$= (-1\cdot)(-1)\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

$$= 1\cdot 2\cdot 3\cdot 4$$

$$= 24.$$

(i) $\det(2A) = 2^4 \cdot \det(A)$

$= 2^4 \cdot 5$

(ii) $\det(A^{-1}B^2C^T)$

$= \det(A^{-1}) \cdot \det(B^2) \cdot \det(C^T)$

$= \dfrac{1}{\det(A)} \cdot \det(B)^2 \cdot \det(C)$

$= \dfrac{1}{5} \cdot 1^2 \cdot 24$

$= \dfrac{24}{5}$

#5.

(a) $W = \{ y : y'' + \cos(x) y' - y = 0 \}$

is a subspace of $F(\mathbb{R})$

Closure under addition: Let y_1 and y_2 be in W.

Then

$$y_1'' + \cos(x) y_1' - y_1 = 0, \text{ and}$$

$$y_2'' + \cos(x) y_2' - y_2 = 0 .$$

So that

$$(y_1 + y_2)'' + \cos(x)(y_1 + y_2)' - (y_1 + y_2)$$

$$= y_1'' + y_2'' + \cos(x)(y_1' + y_2') - y_1 - y_2$$

$$= y_1'' + y_2'' + \cos(x) y_1' + \cos(x) y_2' - y_1 - y_2$$

$$= \left[y_1'' + \cos(x) y_1' - y_1 \right] + \left[y_2'' + \cos(x) y_2' - y_2 \right]$$

$$= 0 + 0$$

$$= 0 .$$

Hence, $y_1 + y_2$ is in W.

Closure under scalar multiplication:

Let $y \in W$ and $c \in \mathbb{R}$. Then

$$y'' + \cos(x) y' - y = 0.$$

So that

$$(cy)'' + \cos(x)(cy)' - (cy)$$

$$= c y'' + \cos(x) \cdot c y' - c y$$

$$= c\left(y'' + \cos(x) y' - y\right)$$

$$= c \cdot 0$$

$$= 0$$

Hence, $cy \in W$.

(b) The set

$$W = \{ v \in \mathbb{R}^n : Av = \lambda v \}$$

is a subspace of $\mathbb{R}^n$.

Closure under addition:

Let $v, w \in W$. Then

$$Av = \lambda v \text{ and } Aw = \lambda w$$

so that

$$A(v+w) = Av + Aw$$

$$= \lambda v + \lambda w$$

$$= \lambda (v+w)$$

Hence, $v+w \in W$.

Closure under scalar multiplication:

Let $v \in W$ and $c \in \mathbb{R}$. Then

$$Av = \lambda v.$$

So that

$$\begin{aligned} A(cv) &= c(Av) \\ &= c(\lambda v) \\ &= (c\lambda)v \\ &= (\lambda c)v \\ &= \lambda(cv) \end{aligned}$$

Hence, $cv \in W$.

(c) The set

$$W = \{ A \in M_3(\mathbb{R}) : A \text{ is invertible} \}$$

is <u>not</u> a subspace of $M_3(\mathbb{R})$.

<u>Note</u>: You can show it is not closed under addition

or it is not closed under scalar multiplication.

I will give an example of both.

Not closed under addition:

The matrices I_3 and $-I_3$ are diagonal matrices w/ determinants 1 and -1 (resp.), hence, they are invertible $(\neq 0)$.

So that I_3 and $-I_3$ are in W.

However,

$$I_3 + (-I_3) = O_3$$

has determinant zero, so is not invertible and not in W.

Not closed under scalar multiplication.

Again the matrix I_3 is in W.

But for $c=0$,

$$c \cdot I_3 = 0 \cdot I_3 = O_3$$

is not invertible so W is not closed under scalar multiplication.

#6.

(a) Suppose c_1, c_2, c_3 are scalars such that

$$c_1\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2\begin{bmatrix}3\\2\\1\end{bmatrix} + c_3\begin{bmatrix}6\\5\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Longleftrightarrow \begin{bmatrix}1 & 3 & 6\\1 & 2 & 5\\1 & 1 & 4\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$ has a unique solution.

$$A = \begin{bmatrix}1 & 3 & 6\\1 & 2 & 5\\1 & 1 & 4\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}1 & 0 & 3\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

free

the system has infinitely many solutions.

$\Rightarrow$ the vectors are not L.I.

(c) not a basis.

(b) $\text{rank}(A) = 2 \rightarrow$ cannot span $\mathbb{R}^3$.

#7

form

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 0 \\ 1 & 3 & 7 & 0 \\ 1 & 4 & 10 & 2 \end{bmatrix}$$

$$\Longrightarrow A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 7 & 10 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$\Longrightarrow \mathrm{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Longrightarrow$ A basis for the vectors is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}$$

#8.

(a) The map $T: C[0,1] \to \mathbb{R}$ given by

$$T(f) = \int_0^1 f(x)e^x\,dx$$

is a linear transformation.

Additivity: Let $f, g \in C[0,1]$, then

$$T(f+g) = \int_0^1 (f+g)(x)e^x\,dx$$

$$= \int_0^1 [f(x) + g(x)]e^x\,dx$$

$$= \int_0^1 f(x)e^x + g(x)e^x\,dx$$

$$= \int_0^1 f(x)e^x\,dx + \int_0^1 g(x)e^x\,dx$$

$$= T(f) + T(g).$$

Homogeneity: Let $f \in C[0,1]$ and $c \in \mathbb{R}$.

Then

$$T(cf) = \int_0^1 (cf)(x) e^x \, dx$$

$$= \int_0^1 c \cdot f(x) e^x \, dx$$

$$= c \cdot \int_0^1 f(x) e^x \, dx$$

$$= c \cdot T(f).$$

(b) The map $T : C^\infty(\mathbb{R}) \to F(\mathbb{R})$

given by

$$T(f)(x) = f''(x) - 2f'(x) + f(x) + 1$$

is <u>not</u> a linear transformation.

Fails Homogeneity: Let $f(x) = e^x$ and $c = 0$.

Then $(cf)(x) = 0 \cdot e^x = 0$ ~~is~~ $\nRightarrow$ so

cf is the zero function

So

$$\left[T(cf)\right](x) = 0 - 2 \cdot 0 + 0 + 1$$

$$= 1$$

$\Rightarrow$ $T(cf)$ is the constant 1 function.

On the other hand,

$$\left[c \cdot T(f)\right](x) = c \cdot \left(T(f)(x)\right)$$

$$= 0 \cdot \left(T(f)(x)\right)$$

$$= 0$$

So $c \cdot T(f)$ is the zero function.

$\therefore$ $T(cf) \neq c \cdot T(f)$

(c) The map $T: P_2 \to P_2$ given by

$$T(ax^2+bx+c) = bx^2+cx+a$$

is a linear transformation.

Additivity: Let $P_1(x) = a_1x^2 + b_1x + c_1$ and $P_2(x) = a_2x^2 + b_2x + c_2$ be polynomials in P_2. Then

$$
\begin{aligned}
&T(P_1(x) + P_2(x)) \\
&= T((a_1x^2+b_1x+c_1) + (a_2x^2+b_2x+c_2)) \\
&= T((a_1+a_2)x^2 + (b_1+b_2)x + (c_1+c_2)) \\
&= (b_1+b_2)x^2 + (c_1+c_2)x + (a_1+a_2) \\
&= (b_1x^2 + c_1x + a_1) + (b_2x^2 + c_2x + a_2) \\
&= T(a_1x^2 + b_1x + c_1) + T(a_2x^2 + b_2x + c_2) \\
&= T(P_1(x)) + T(P_2(x))
\end{aligned}
$$

Homogeneity: Let $p(x) = ax^2 + bx + c$ be in P_2

and let $k \in \mathbb{R}$. Then

$$T(k \cdot p(x))$$

$$= T(k \cdot (ax^2 + bx + c))$$

$$= T((ka)x^2 + (kb)x + (kc))$$

$$= (kb)x^2 + (kc)x + (ka)$$

$$= k(bx^2) + k(cx) + ka$$

$$= k(bx^2 + cx + a)$$

$$= k \cdot T(ax^2 + bx + c)$$

$$= k \cdot T(p(x))$$

#9. The characteristic poly. is

$$p(x) = (x-2)^3$$

root: 2 (mult. 3)

$\Longrightarrow$ $y_1 = e^{2x}$, $y_2 = xe^{2x}$, $y_3 = x^2 e^{2x}$

is a candidate for a basis of ker(L)

Claim: y_1, y_2, y_3 are L.I.

$$W(y_1, y_2, y_3)$$

$$= W(e^{2x}, e^{2x}\cdot x, e^{2x}\cdot x^2)$$

$$= (e^{2x})^3 \cdot W(1, x, x^2)$$

$$= e^{6x} \cdot \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}$$

$$= 2 \cdot e^{6x}$$

$\neq 0$ everywhere. $\Longrightarrow$ y_1, y_2, y_3 are L.I.

Since nullity$(L) = 3$, then y_1, y_2, y_3 form a basis for $\ker(L)$

#10.

$$p(\lambda) = \begin{vmatrix} \lambda+7 & -6 & 6 \\ 3 & \lambda-2 & 3 \\ -6 & 6 & \lambda-5 \end{vmatrix}$$

$$= (\lambda+7)\left[(\lambda-2)(\lambda-5) - 18\right] + 6\left[3(\lambda-5)+18\right] + 6\left[18 + 6(\lambda-2)\right]$$

$$= (\lambda+7)\left[\lambda^2 - 7\lambda - 8\right] + 18\left[\lambda - 5 + 6\right] + 36\left[\cancel{18}\,3 + \lambda - 2\right]$$

$$= (\lambda+7)(\lambda-8)(\lambda+1) + 18(\lambda+1) + 36(\lambda+1)$$

$$= (\lambda+1)\left[\lambda^2 - \lambda - 56 + 18 + 36\right]$$

$$= (\lambda+1)\left[\lambda^2 - \lambda + 2\right]$$ $\longrightarrow$ next page

minus

~~discriminant $b^2 - 4ac = 1 - 4\cdot1\cdot2 < 0$~~

~~imaginary roots~~

$$= (\lambda+1)(\lambda-2)(\lambda+1)$$

$$= (\lambda+1)^2(\lambda-2)$$

eigenvalues : $\lambda = -1$ (mult 2)

$\lambda = 2$ (mult 1)

$\lambda = -1$:

$$\left[\begin{array}{ccc|c} 6 & -6 & 6 & 0 \\ 3 & -3 & 3 & 0 \\ -6 & 6 & -6 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} d & f & f & \end{array} \end{array}$$

y, z free and $x = y - z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_{-1} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y-z \\ y \\ z \end{bmatrix}$$

$$= y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

basis for E_{-1}

$\lambda = 2$:

$$\begin{array}{l}\left[\begin{array}{ccc|c} 9 & -6 & 6 & 0 \\ 3 & 0 & 3 & 0 \\ -6 & 6 & -3 & 0 \end{array}\right]\end{array} \begin{array}{l} R_1/3 \\ R_2/3 \\ R_3/3 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 3 & -2 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 2 & -1 & 0 \end{array}\right] \begin{array}{l} \\ 3R_2 - R_1 \\ 3R_3 + 2R_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 3 & -2 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\longrightarrow \begin{array}{c} \left[\begin{array}{ccc|c} 3 & 0 & 3 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} d & d & f & \end{array} \end{array}$$

z free, $y = -\frac{1}{2}z$, $x = -z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_2 \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -\frac{1}{2}z \\ z \end{bmatrix} = \frac{z}{2}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

basis for E_2

(c) $$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

#11.

(b)

$$P = [I]_\beta^\alpha = \left[[1]_\alpha \; [x-2]_\alpha \; [\overbrace{x^2-4x+4}^{(x-2)^2}]_\alpha \right]$$

$$= \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

Note: the order in which the vectors in α & β appear matters

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{RREF} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

(c)

$$[T]_\alpha^\alpha = \left[[T(1)]_\alpha \; [T(x-2)]_\alpha \; [T((x-2)^2)]_\alpha \right]$$

$f(x) = 1$: $\quad f'(x) = 0.$

$$T(f)(x) = f(2) + f'(2)(x-2)$$
$$= 1 + 0(x-2)$$
$$= 1$$

- $f(x) = x$: $f'(x) = 1$

$$
\begin{aligned}
T(f)(x) &= f(2) + f'(2)(x-2) \\
&= 2 + 1(x-2) \\
&= 2 + x - 2 \\
&= x
\end{aligned}
$$

- $f(x) = x^2$: $f'(x) = 2x$

$$
\begin{aligned}
T(f)(x) &= f(2) + f'(2)(x-2) \\
&= 2^2 + 2 \cdot 2 (x-2) \\
&= 4 + 4x - 8 \\
&= 4x - 4
\end{aligned}
$$

So $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

(d)

$$[T]_\beta^\beta = [I]_\alpha^\beta [T]_\alpha^\alpha [I]_\beta^\alpha$$

$$= P^{-1} [T]_\alpha^\alpha P$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(e) $[2x^2 - x + 1]_\beta = [I(2x^2 - x + 1)]_\beta$

$$= [I]_\alpha^\beta [2x^2 - x + 1]_\alpha$$

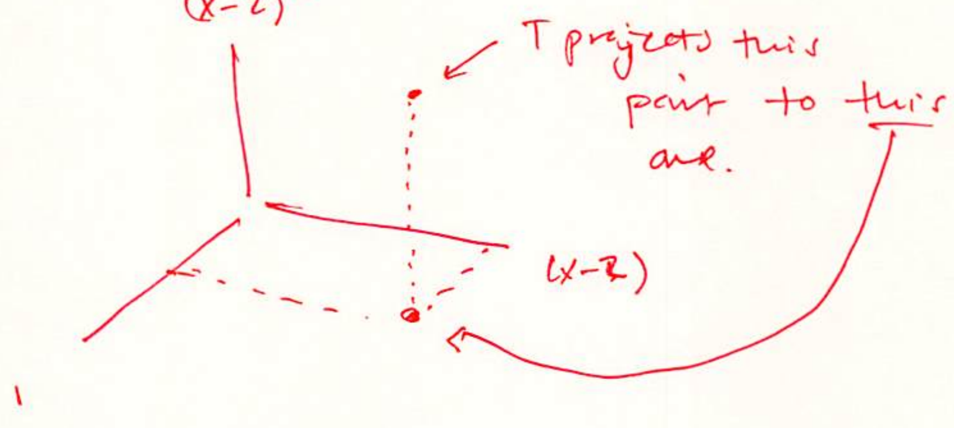
P^{-1}

$$= \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 7 \\ 2 \end{bmatrix}$$

(7) T projects onto a plane.

If we view P_2 as a 3-D space



#12.

(i) solve $Y' = DY$

$$Y' = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} Y$$

has solution (fundamental)

$$z_1 = \begin{bmatrix} e^{-x} \\ 0 \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} 0 \\ e^{-x} \\ 0 \end{bmatrix}, \quad z_3 = \begin{bmatrix} 0 \\ 0 \\ e^{2x} \end{bmatrix}$$

(2) Solve $Y' = AY$.

Then

$$Y_1 = PZ_1 = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-x} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-x} \\ e^{-x} \\ 0 \end{bmatrix}$$

$$Y_2 = PZ_2 = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ e^{-x} \\ 0 \end{bmatrix} = \begin{bmatrix} -e^{-x} \\ 0 \\ e^{-x} \end{bmatrix}$$

$$Y_3 = PZ_3 = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ e^{2x} \end{bmatrix} = \begin{bmatrix} -2e^{2x} \\ -e^{2x} \\ 2e^{2x} \end{bmatrix}$$

is a fundamental set of solutions for $Y' = AY$.

So

General solution: $$Y_H = \begin{bmatrix} c_1 e^{-x} - c_2 e^{-x} - 2c_3 e^{2x} \\ c_1 e^{-x} - c_3 e^{2x} \\ c_2 e^{-x} + 2c_3 e^{2x} \end{bmatrix}$$

Matrix of fundamental solutions: $$M = \begin{bmatrix} e^{-x} & -e^{-x} & -2e^{2x} \\ e^{-x} & 0 & -e^{2x} \\ 0 & e^{-x} & 2e^{2x} \end{bmatrix}$$

(3) Find particular solution $Y_p = M\int M^{-1}G(x)\,dx$

(i) Find M^{-1}:

$$\left[\begin{array}{ccc|ccc} e^{-x} & -e^{-x} & -2e^{2x} & 1 & 0 & 0 \\ e^{-x} & 0 & -2e^{2x} & 0 & 1 & 0 \\ 0 & e^{-x} & 2e^{2x} & 0 & 0 & 1 \end{array}\right] R_2 - R_1$$

$$\rightarrow \left[\begin{array}{ccc|ccc} e^{-x} & -e^{-x} & -2e^{2x} & 1 & 0 & 0 \\ 0 & e^{-x} & 0 & -1 & 1 & 0 \\ 0 & e^{-x} & 2e^{2x} & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 + R_2 \\ \\ R_3 - R_2 \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} e^{-x} & 0 & -2e^{2x} & 0 & 1 & 0 \\ 0 & e^{-x} & 0 & -1 & 1 & 0 \\ 0 & 0 & 2e^{2x} & 1 & -1 & 1 \end{array}\right] R_1 + R_3$$

$$\rightarrow \left[\begin{array}{ccc|ccc} e^{-x} & 0 & 0 & 1 & 0 & 1 \\ 0 & e^{-x} & 0 & -1 & 1 & 0 \\ 0 & 0 & 2e^{2x} & 1 & -1 & 1 \end{array}\right] \begin{array}{l} R_1/e^{-x} \\ R_2/e^{-x} \\ R_3/2e^{2x} \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & e^{+x} & 0 & e^{+x} \\ 0 & 1 & 0 & -e^{x} & e^{x} & 0 \\ 0 & 0 & 1 & \frac{1}{2}e^{-2x} & -\frac{1}{2}e^{-2x} & \frac{1}{2}e^{-2x} \end{array}\right]$$

$$\Rightarrow M^{-1} = \begin{bmatrix} e^x & 0 & e^x \\ -e^x & e^x & 0 \\ \frac{1}{2}e^{-2x} & -\frac{1}{2}e^{2x} & \frac{1}{2}e^{2x} \end{bmatrix}$$

(ii) Find $M^{-1}G(x)$

$$\begin{bmatrix} e^x & 0 & e^x \\ -e^x & e^{\cancel{2}x} & 0 \\ \frac{1}{2}e^{-2x} & -\frac{1}{2}e^{-2x} & \frac{1}{2}e^{-2x} \end{bmatrix} \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ xe^x \\ -\frac{1}{2}e^{-2x} \end{bmatrix}$$

(iii) Find $\int M^{-1}G(x)\,dx$

$$\int xe^x = xe^x - e^x$$

$x \quad + \quad e^x$

$1 \quad - \quad e^x$

$0 \quad \quad e^x$

$$\int -\tfrac{1}{2}e^{-2x} = +\frac{1}{4}e^{-2x}$$

$$\Longrightarrow \int M^{-1}G(x)\,dx = \begin{bmatrix} 0 \\ xe^x - e^x \\ +\frac{1}{4}e^{-2x} \end{bmatrix}$$

(iv) Find $Y_p = M \int M^{-1} G(x)\, dx$

$$Y_p = \begin{bmatrix} e^{-x} & -e^{-x} & -2e^{2x} \\ e^{-x} & 0 & -2e^{2x} \\ 0 & e^{-x} & 2e^{2x} \end{bmatrix} \begin{bmatrix} e \\ xe^{x} - e^{x} \\ 1/4\, e^{-2x} \end{bmatrix}$$

$$= \begin{bmatrix} (1-x) - 1/2 \\ -1/2 \\ (x-1) + 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 - x \\ -1/2 \\ x - 1/2 \end{bmatrix}$$

$\therefore$ The solution to $Y' = AY + G(x)$ is

$$Y = Y_H + Y_p$$

$$= \begin{bmatrix} (c_1 - c_2)e^{-x} - 2c_3 e^{2x} + 1/2 - x \\ c_1 e^{-x} - c_3 e^{2x} - 1/2 \\ c_2 e^{-x} + 2c_3 e^{2x} + x - 1/2 \end{bmatrix}$$

#13.

(1) Solve $Y' = AY$

$$y_1' = 2y_1 + y_2$$
$$y_2' = 2y_2$$
$$y_3' = 3y_3$$

Immediately get solutions for 2nd two eqn's

$$y_2 = C_2 e^{2x} \quad \& \quad y_3 = C_3 e^{3x}$$

Back-sub into 1st eqn.

$$y_1' = 2y_1 + C_2 e^{2x}$$

$$\Longrightarrow y_1' \underbrace{- 2y_1}_{P(x)} = C_2 e^{2x}$$

$$I = e^{\int P(x)\,dx} = e^{\int -2\,dx} = e^{-2x}$$

$$\Longrightarrow y_1 = e^{2x} \int e^{-2x} \cdot C_2 e^{2x}\,dx$$

$$= e^{2x} \int C_2\,dx = e^{2x}(C_2 x + C_1)$$

Claim:

$$\mathbf{Z}_H = \begin{bmatrix} c_1 e^{2x} + c_2 x e^{2x} \\ c_2 e^{2x} \\ c_3 e^{3x} \end{bmatrix}$$

$$= c_1 \underbrace{\begin{bmatrix} e^{2x} \\ 0 \\ 0 \end{bmatrix}}_{Z_1} + c_2 \underbrace{\begin{bmatrix} x e^{2x} \\ e^{2x} \\ 0 \end{bmatrix}}_{Z_2} + c_3 \underbrace{\begin{bmatrix} 0 \\ 0 \\ e^{3x} \end{bmatrix}}_{Z_3}$$

is the general solution to $Y' = JY$.

$\ulcorner$ Since the space of solutions to $Y' = JY$ has dimension 3, it suffices to prove Z_1, Z_2, Z_3 are L.I.

$$W(Z_1, Z_2, Z_3) = \begin{vmatrix} e^{2x} & x e^{2x} & 0 \\ 0 & e^{2x} & 0 \\ 0 & 0 & e^{3x} \end{vmatrix}$$

upper triangular.

$$= e^{7x} \neq 0$$

everywhere.

$\therefore$ L.I

$\llcorner$

(2) Solve $Y' = AY$.

Then the general solution to $Y' = AY$

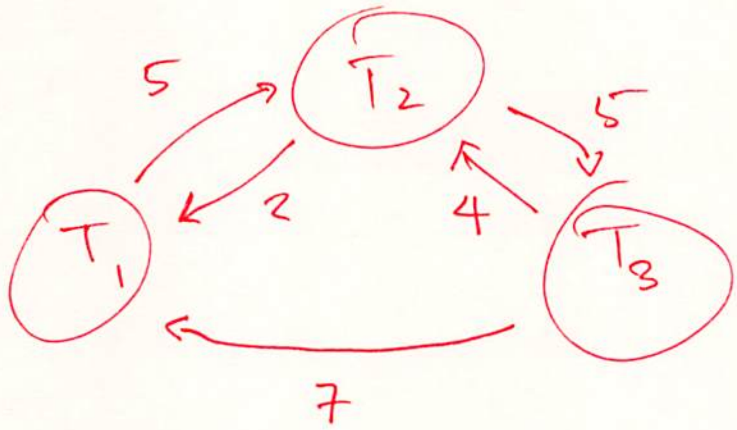
$$Y_H = P Z_H$$

$$= \begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{2x} + c_2 x e^{2x} \\ c_2 e^{2x} \\ c_3 e^{3x} \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{2x} + c_2 x e^{2x} - 2c_2 e^{2x} - c_3 e^{3x} \\ 2c_2 e^{2x} + c_3 e^{3x} \\ -c_1 e^{2x} - c_2 x e^{2x} + c_2 e^{2x} + c_3 e^{3x} \end{bmatrix}$$

#14.

(a)



(b) sum of a row i = flow out of tank i

sum of a column j = flow into column j

(c) $\Longrightarrow$ net flow into T_1 $= (2+7) - 5 = 4$ L/min

net flow into T_2 $= (5+4) - (2+5) = 2$ L/min

net flow into T_3 $= (5) - (7+4) = -6$ L/min

Volume of tank T_1 at time t $= 10000 + 4t$

Volume of tank T_2 at time t $= 10000 + 2t$

Volume of tank T_3 at time t $= 10000 - 6t$

Let y_1, y_2, y_3 be the amnt of salt in T_1, T_2, T_3 at time t (resp.)

Concentration of T_1 $= \dfrac{y_1}{10000 + 4t}$

Concentration of T_2 $= \dfrac{y_2}{10000 + 2t}$

Concentration of T_3 $= \dfrac{y_3}{10000 - 6t}$

$$y_1' = \text{Rate into } T_1 - \text{Rate out of } T_1$$

$$= 2 \cdot \frac{y_2}{10000+2t} + 7 \frac{y_3}{10000-6t} - 5 \frac{y_1}{10000+4t}$$

$$y_2' = 5 \frac{y_1}{10000+4t} + 4 \frac{y_3}{10000-6t} - 7 \cdot \frac{y_2}{10000+2t}$$

$$y_3' = 5 \frac{y_2}{10000+2t} - 11 \cdot \frac{y_3}{10000-6t}$$

(d)

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -\frac{5}{10000+4t} & \frac{2}{10000+2t} & \frac{7}{10000-6t} \\ \frac{5}{10000+4t} & \frac{-7}{10000+2t} & \frac{4}{10000-6t} \\ 0 & \frac{5}{10000+2t} & \frac{-11}{10000-6t} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

#15.

(a)

$$F = ma$$

$$\Longrightarrow \quad a = \frac{1}{m} F$$

$$\Longrightarrow \quad \begin{bmatrix} d^2x/dt^2 \\ d^2y/dt^2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 120x - 56y \\ -6x - 24y \end{bmatrix}$$

$$= \begin{bmatrix} 20x - \frac{28}{3}y \\ -x - 4y \end{bmatrix}$$

(b) Let

$$y_1 = x$$

$$y_2 = x'$$

$$y_3 = y$$

$$y_4 = y'$$

$$v_1' = x' = v_2$$

$$v_2' = x'' = 20x - \tfrac{28}{3}y = 20v_1 - \tfrac{28}{3}v_3$$

$$v_3' = y' = v_4$$

$$v_4' = y'' = -x - 4y = -v_1 - 4v_3$$

$$\begin{bmatrix} v_1' \\ v_2' \\ v_3' \\ v_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 20 & 0 & -28/3 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -4 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

(c) entries 1 & 3.

#16. Recall the operator was $L = (D-2)^3$

(a)
$$
\begin{aligned}
y \in \ker(L) &\iff Ly = 0 \\
&\iff (D-2)^3 y = 0 \\
&\iff \left(D^3 - 6D^2 + 12D - 8\right) y = 0 \\
&\iff \underbrace{y^{(3)} - 6y'' + 12y' - 8y = 0}_{\text{this diffeq.}}
\end{aligned}
$$

(b) Let

$$
\begin{cases}
v_1 = y \\
v_2 = y' \\
v_3 = y''
\end{cases}
$$

$$
\Longrightarrow \quad
\begin{aligned}
v_1' &= y' = v_2 \\
v_2' &= y'' = v_3 \\
v_3' &= y''' = 8y - 12y' + 6y'' \\
&= 8v_1 - 12v_2 + 6v_3
\end{aligned}
$$

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -12 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

(c) the first entry.

#17. Let

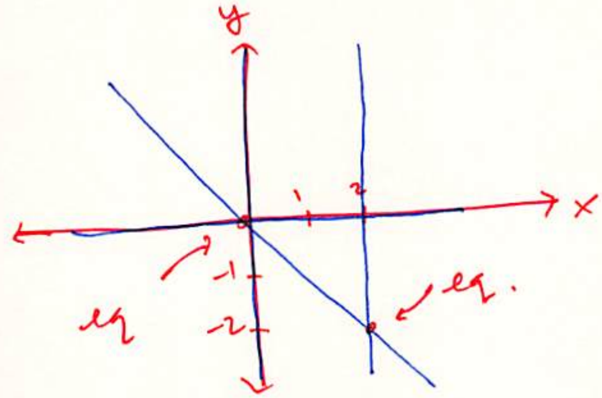
$$F(x,y) = xy - 2y$$
$$G(x,y) = 2x^2 + xy$$

Suppose $x = x_0$ & $y = y_0$ is an equilibrium solution

$$\Rightarrow x' = 0 \text{ and } y' = 0$$

$$\Rightarrow 0 = x' = F(x,y) = xy - 2y = y(x-2)$$
$$0 = y' = G(x,y) = 2x^2 + xy = x(2x+y)$$

$$\Rightarrow \quad y = 0 \text{ or } x = 2$$
and
$$x = 0 \text{ or } y = -2x$$



The equilibrium solutions are $(0,0)$ and $(2,-2)$

Linear part at $(0,0)$:

$F_x(x,y) = y \Rightarrow F_x(0,0) = 0$

$F_y(x,y) = x-2 \Rightarrow F_y(0,0) = -2$

$G_x(x,y) = 4x+y \Rightarrow G_x(0,0) = 0$

$G_y(x,y) = x \longrightarrow G_y(0,0) = 0$

Taylor rep's at $(0,0)$

$$F(x,y) = F(0,0) + F_x(0,0)(x-0) + F_y(0,0)(y-0) + R_F(x-0,y-0)$$

$$= 0 + 0 + (-2)(x-0) + R_F(x-0,y-0)$$

$$= -2x + R_F(x,y)$$

$$G(x,y) = G(0,0) + G_x(0,0)(x-0) + G_y(0,0)(y-0) + R_G(x-0,y-0)$$

$$= 0 + 0 + 0 + R_G(x,y)$$

$$= R_G(x,y)$$

Then

$$x' = F(x,y) = -2x + R_F(x,y)$$

$$y' = G(x,y) = R_G(x,y)$$

$$\Rightarrow \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}_{\text{Linear part at } (0,0).} + \begin{bmatrix} R_F(x,y) \\ R_G(x,y) \end{bmatrix}$$

Linear part at $(2,-2)$:

$F_x(2,-2) = -2$

$F_y(2,-2) = 2-2 = 0$

$G_x(2,-2) = 8-2 = 6$

$G_y(2,-2) = 2$

Taylor expansions at $(2,-2)$:

$$\begin{cases} F(x,y) = F(2,-2) + F_x(2,-2)(x-2) + F_y(2,-2)(y+2) + R_F(x-2,y+2) \\ G(x,y) = G(2,-2) + G_x(2,-2)(x-2) + G_y(2,-2)(y+2) + R_G(x-2,y+2) \end{cases}$$

$$\Rightarrow \begin{cases} F(x,y) = -2(x-2) + R_F(x-2,y+2) \\ G(x,y) = 6(x-2) + 2(y+2) + R_G(x-2,y+2) \end{cases}$$

Substitute $u = x-2 \Rightarrow u' = x'$

$v = y+2 \qquad v' = y'$

$$u' = x' = F(x,y) = -2u + R_F(u,v)$$

$$v' = y' = G(x,y) = 6u + 2v + R_G(u,v)$$

$$\Longrightarrow \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} R_F(u,v) \\ R_G(u,v) \end{bmatrix}$$

Linear part at $(3,-2)$