Math 307 Final Practice, Spring 2023

Name:

Question	Points	Score
1	0	
2	0	
3	0	
4	0	
5	0	
6	0	
7	0	
8	0	
9	0	
10	0	
11	0	
12	0	
13	0	
14	0	
15	0	
16	0	
17	0	
Total:	0	

- You have 120 minutes to complete this exam.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work, unless otherwise indicated. You will get almost no credit for solutions that are not fully justified.
- No electronic devices are authorized with the exception of a scientific calculator.
- The back side of each page can be used as scratch work or for longer solutions. Please indicate clearly if your solution extends to the back side of a page.

1. True/False questions. No justification necessary.

(a) True False If A is a 3×5 matrix, then there is always a nonzero vector $x \in \mathbb{R}^5$ such that $Ax = 0$.

(b) True False There is an 2×2 upper triangular matrix A such that

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(c) True False Every invertible matrix is a product of elementary matrices.

(d) True False If $\det(A) = \det(A^T)$, then A must be symmetric.

(e) True False There is a real vector space with exactly 7 vectors.

(f) True False If U and W are subpaces of a vector space V, then the space of all sums

$$U + W := \{u + w : u \in U \text{ and } w \in W\}$$

is a subspace of V.

(g) True False The columns of any 3×4 matrix must be linearly dependent.

(h) True False Every basis of $M_{3\times 4}(\mathbb{R})$ has 12 matrices.

(i) True False If v_1, v_2, v_3 is a basis for V, then

$$v_1, v_1 + v_2, v_1 + v_2 + v_3, v_1 + v_3,$$

is also a basis for V.

(j) True False If $W(y_1, y_2, y_3) = 0$ on (a, b), then y_1, y_2, y_3 are linearly dependent in $F(a, b)$.

(k) True False Every polynomial is a linear transformation from $\mathbb{R}$ to $\mathbb{R}$.

(l) True False If q is any polynomial, then

$$e^{-2x}, xe^{-2x}, x^2e^{-2x}, x^3e^{-2x}$$

are all in the kernel of the differential operator $q(D)(D + 2)^4$.

(m) True False Every linear transformation from $T : R^n \to \mathbb{R}^m$ is a matrix transformation $Tx = Ax$ for some matrix $A \in M_{m\times n}(\mathbb{R})$.

(n) True False If v is an eigenvector with eigenvalue 2, then $7v$ is an eigenvector with eigenvalue 14.

(o) True False If A and B are similar, then $\det(A) = \det(B)$.

(p) True False If A is diagonalizable, then $\det(A)$ is equal to the product of its eigenvalues.

(q) True False The matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

is in Jordan canonical form.

(r) True False Every matrix is similar to an upper triangular matrix.

(s) True False The dimension of the space of solutions for a homogeneous system of n first-order linear differential equations is n.

2. Solve the system of linear equations

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 - x_4 &= 4 \\ 5x_1 + 4x_2 + 3x_3 - x_4 &= 4 \\ -2x_1 - 2x_2 - x_3 + 2x_4 &= -3 \\ 11x_1 + 6x_2 + 4x_3 + x_4 &= 11 \end{aligned}$$

3. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Find $(AB)^T$, BA, $(BC)^{-1}$, CB, A^{-1}, B^{-1}, $B^T + 2C$. or state that the expression is undefined.

4. Let

$$A = \begin{bmatrix} -2 & 1 & 5 & 2 \\ -3 & -1 & 0 & -1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Find $\det(2A)$ and $\det(A^{-1}B^2C^T)$.

5. Prove or disprove the following.
   (a) The set of solutions of the differential equation $y'' + \cos(x)y' - y = 0$ is a subspace of $F(\mathbb{R})$.
   (b) The set of vectors satisfying $Av = \lambda v$ for a fixed $n \times n$ matrix A is a subspace of $\mathbb{R}^n$.
   (c) The set of invertible 3×3 matrices is a subspace of $M_3(\mathbb{R})$.

6. Consider the vectors

$$\alpha = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \right\}$$

   (a) Are the vectors in α linearly independent set of vectors? Clearly justify.
   (b) Do the vectors in α span $\mathbb{R}^3$? Clearly justify.
   (c) Does α form a basis of $\mathbb{R}^3$?

7. Find a basis for the span of the vectors

$$\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}1\\4\\7\\10\end{bmatrix}, \begin{bmatrix}2\\0\\0\\2\end{bmatrix}$$

8. Prove or disprove the following.
   (a) The map $T : C[0,1] \to \mathbb{R}$ given by

   $$T(f) = \int_0^1 f(x)e^x \, dx$$

   is a linear transformation.
   (b) The map $T : C^\infty(\mathbb{R}) \to F(\mathbb{R})$ given by

   $$T(f)(x) = f''(x) - 2f'(x) + f(x) + 1,$$

   is a linear transformation.
   (c) The map $T : P_2 \to P_2$ given by

   $$T(ax^2 + bx + c) = bx^2 + cx + a,$$

   is a linear transformation.

9. Find a basis for the 3rd order linear differential operator $L = (D-2)^3$.

10. Consider the following matrix

$$A = \begin{bmatrix}-7 & 6 & -6\\-3 & 2 & -3\\6 & -6 & 5\end{bmatrix}$$

   (a) Find the eigenvalues of A.
   (b) Find bases for the eigenspaces of A.
   (c) Is A diagonalizable? If so, give matrices P and D such that $D = P^{-1}AP$.

11. Let $T : P_2 \to P_2$ be the linear transformation $T(f)(x) = f(2) + f'(2)(x-2)$. That is T maps each f to its Taylor polynomial of order 1 about $x = 2$.

   Let

$$\alpha = \{1, x, x^2\},$$
$$\beta = \{1, (x-2), (x-2)^2\}.$$

   (a) Make sure you know how to prove α and β are bases of P_2, and that T is a linear transformation.

(b) Find the change of basis matrix $P = [I]_\beta^\alpha$ and its inverse.

(c) Find the matrix of T with respect to α.

(d) Find the matrix of T with respect to β.

(e) Find $[2x^2 - x + 1]_\beta$.

(f) Describe what T does geometrically to P_2.

12. Find the general solution to the nonhomogeneous system of first-order linear equations

$$Y' = AY + \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix}.$$

Where A is the matrix from problem 10.

13. The Jordan decomposition of the matrix

$$A = \begin{bmatrix} -1 & -1 & -3 \\ 2 & 3 & 2 \\ 3 & 1 & 5 \end{bmatrix}$$

is $J = P^{-1}AP$ where

$$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Find the general solution to system $Y' = AY$.

14. Suppose there are three tanks labeled 1,2 and 3 with pipes pumping well-mixed solution between them. Tank 1 originally starts with 10kg of salt, and each tank begins with 10000 liters of water. Consider the adjacency matrix

$$A = \begin{bmatrix} 0 & 5 & 0 \\ 2 & 0 & 5 \\ 7 & 4 & 0 \end{bmatrix}.$$

Each entry a_{ij} represents the flow of solution from tank i to tank j. For instance, the entry $a_{12} = 5$ means 5 liters of solution are piped from tank 1 into tank 2 per second. Express

(a) Sketch the tanks, pipes, and label each pipe with its flow of solution.

(b) What does the sum of the entries in a row represent? What about the sum of the entries in a column?

(c) Write a system of first order differential equations that describes the amount of salt in each tank at time t.

(d) Write your system in matrix form.

15. A force

$$F = \begin{bmatrix} 120x - 56y \\ -6x - 24y \end{bmatrix}$$

is applied to a mass of 6kg.

   (a) Use Newton's second law of motion $F = ma$ to obtain a system of two second-order linear differential equations that describe the position of the particle at time t.

   (b) Convert the system in part (a) to a system of four first-order linear differential equations.

   (c) If Y_H is the general solution to the system in (b), which entries describe the position of the mass?

16. The kernel of the operator in problem 9 is the same as the set of solutions to a third order linear differential equation.

   (a) Which differential equation?

   (b) Convert the differential equation you found (a) into a system of *first* order linear differential equations.

   (c) If Y_H is the general solution to your system in (b), which entry is the general solution to the equation in (a).

17. Consider the 2×2 system

$$\begin{aligned} x' &= xy - 2y \\ y' &= 2x^2 + xy \end{aligned}$$

   (a) Find the equilibrium solutions.

   (b) Find the linear part at each of the equilibrium solutions.