2.1

#7.

$$V = \{ \{a_n\} : \text{the sequence } \{a_n\} \text{ converges} \}$$

Addition: $\{a_n\} + \{b_n\} := \{a_n + b_n\}$

scalar mult: $c \cdot \{a_n\} := \{ca_n\}$

(A1) Let $\{a_n\}, \{b_n\} \in V$, then

$$\begin{aligned} \{a_n\} + \{b_n\} &= \{a_n + b_n\} \\ &= \{b_n + a_n\} \\ &= \{b_n\} + \{a_n\} \end{aligned}$$

(A2) Let $\{a_n\}, \{b_n\}, \{c_n\} \in V$, then

$$\begin{aligned} \{a_n\} + (\{b_n\} + \{c_n\}) &= \{a_n\} + \{b_n + c_n\} \\ &= \{a_n + (b_n + c_n)\} \\ &= \{(a_n + b_n) + c_n\} \end{aligned}$$

$\longrightarrow$

$$= \{a_n + b_n\} + \{c_n\}$$

$$= (\{a_n\} + \{b_n\}) + \{c_n\}$$

(A3) The zero sequence $\{0\}_{n=1}^{\infty}$ is a convergent sequence, so is in V, and for any $\{a_n\}_{n=1}^{\infty}$ in V we have

$$\{a_n\}_{n=1}^{\infty} + \{0\}_{n=1}^{\infty} = \{a_n + 0\}_{n=1}^{\infty}$$
$$= \{a_n\}_{n=1}^{\infty}.$$

(A4) Let $\{a_n\} \in V$. Then the sequence $\{-a_n\}$ is convergent by calc II., and

$$\{a_n\} + \{-a_n\} = \{a_n - a_n\} = \{0\}_{n=1}^{\infty}$$

(A5) Let $\{a_n\}, \{b_n\} \in V$, and let $c \in \mathbb{R}$.

Then

$$c(\{a_n\} + \{b_n\}) = c\{a_n + b_n\}$$

$$= \{c(a_n + b_n)\}$$

$$= \{ca_n + cb_n\}$$

$$= \{ca_n\} + \{cb_n\}$$

(A6) Let $c, d \in \mathbb{R}$ and $\{a_n\} \in V$, then

$$(c+d)\{a_n\} = \{(c+d)a_n\}$$

$$= \{ca_n + da_n\}$$

$$= \{ca_n\} + \{da_n\}$$

$$= c \cdot \{a_n\} + d \cdot \{a_n\}$$

(A7): Let $c, d \in \mathbb{R}$, and $\{a_n\} \in V$. Then

$$
\begin{aligned}
c\left(d\cdot\{a_n\}\right) &= c\cdot\{da_n\} \\
&= \{c\cdot(da_n)\} \\
&= \{(cd)\,a_n\} \\
&= (cd)\{a_n\}.
\end{aligned}
$$

(A8) ~~The constant 1 sequence~~ $\cancel{\{1\}_{n=1}^{\infty}}$ ~~converges, so is in V, and~~

(A8) For any $\{a_n\} \in V$,

$$1\cdot\{a_n\} = \{1\cdot a_n\} = \{a_n\}.$$

2.2

#4.

(a) $W = \{ \text{diag}(a_1, \dots, a_n) : a_1, \dots, a_n \in \mathbb{R} \}$

(i) Let $A, B \in W$. Then

$A = \text{diag}(a_1, \dots, a_n)$ & $B = \text{diag}(b_1, \dots, b_n)$.

Then

$$A+B = \text{diag}(a_1, \dots, a_n) + \text{diag}(b_1, \dots, b_n)$$

$$= \text{diag}(a_1 + b_1, \dots, a_n + b_n)$$

$$\in W$$

$\Longrightarrow$ W is closed under addition.

(ii) Let $A \in W$ be as above and $c \in \mathbb{R}$, then

$$cA = c \cdot \text{diag}(a_1, \dots, a_n) = \text{diag}(ca_1, \dots, ca_n) \in W$$

$\Longrightarrow$ W is closed under scalar mult.

$\therefore$ W is a subspace.

(b) $W = \{A : A \text{ is upper triangular}\}$

The sum of two upper-triangular matrices is again upper triangular, and scaling an upper triangular matrix remains upper triangular.
Therefore, W is a subspace

(c) $W = \{A : A \text{ is symmetric}\}$

$= \{A : A^T = A\}$

Let $A, B \in W$, then

$$(A+B)^T = A^T + B^T = A + B$$

$\Longrightarrow A + B \in W$

$\Rightarrow W$ is closed under addition

Let $A \in W$ and $c \in \mathbb{R}$, then

$$(cA)^T = c \cdot A^T = c \cdot A$$

$\Longrightarrow c \cdot A \in W$

$\Rightarrow W$ is closed under scalar mult.

$\therefore W$ is a subspace.

(d) $W = \{A : \det A = 0\}$

Let $A = \text{diag}(1, 0, \dots, 0)$ and $B = \text{diag}(0, 1, 1, \dots, 1)$

Then both A and B are in W

(the determinant of a diagonal matrix is the product of its diagonal entries) But

$A + B = I_n$ has determinant $1 \neq 0$.

$\therefore A + B \notin W$

$\therefore W$ is not closed under addition

$\therefore W$ is not a subspace.

(e) $W = \{A : A$ is invertible$\}$.

The matrices $A = I_n$ and $B = -I_n$ are in W

But $A + B = I_n + (-I_n) = O_n$ is not invertible

$\therefore A + B \notin W$

$\therefore W$ is not closed under addition

$\therefore W$ is not a subspace.

4.

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -3 \end{bmatrix}$$

$$\Leftrightarrow \left.\begin{array}{rrrl} c_1 & + c_2 & + 2c_3 & = 1 \\ -c_1 & + c_2 & & = -5 \\ & c_2 & + c_3 & = -3 \end{array}\right\}$$ In span if this system has a solution

Check:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ -1 & 1 & 0 & -5 \\ 0 & 1 & 1 & -3 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$ system is inconsistent

$\therefore$ vector is not in span.