#17. $v_1(x) = x^2+x+1$, $v_2(x) = x^2-x+1$,

$v_3(x) = x^2-1$.

Span: Let $v(x) = ax^2+bx+c$ be arbitrary in $P_2$.

There are scalars $c_1, c_2, c_3$ s.t.

$$c_1v_1 + c_2v_2 + c_3v_3 = v$$

$\iff$ for all $x$

$$(c_1v_1 + c_2v_2 + c_3v_3)(x) = v(x)$$

$\iff$ $c_1 \cdot v_1(x) + c_2 \cdot v_2(x) + c_3 \cdot v_3(x) = v(x)$

$\iff$ $c_1(x^2+x+1) + c_2(x^2-x+1) + c_3(x^2-1) = ax^2+bx+1$

$\iff$ the system

$$\begin{aligned} c_1 + c_2 + c_3 &= a \\ c_1 - c_2 \quad &= b \\ c_1 + c_2 - c_3 &= c \end{aligned}$$

has a solution.

Form the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 1 & -1 & 0 & b \\ 1 & 1 & -1 & c \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & a \\ 0 & -2 & -1 & b-a \\ 0 & 0 & -2 & c-a \end{array}\right]$$

3x3 matrix has 3 leading 1's

$\Longrightarrow$ has a unique solution.

$\therefore$ the vectors $v_1, v_2, v_3$ span $P_2$.

L.I.: with $a=0$, $b=0$, and $c=0$ in above $v(x)=0$. The above argument shows

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{0}$$

has a unique solution, hence, it must be the trivial solution.

$\therefore$ $v_1, v_2, v_3$ are L.I.

#6. Suppose $c_1, c_2, c_3$ are scalars s.t.

$$c_1\begin{bmatrix}0\\4\\-1\end{bmatrix} + c_2\begin{bmatrix}1\\5\\-3\end{bmatrix} + c_3\begin{bmatrix}1\\-3\\-1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Longrightarrow \quad \begin{aligned} c_2 + c_3 &= 0 \\ 4c_1 + 5c_2 - 3c_3 &= 0 \\ -c_1 - 3c_2 - c_3 &= 0 \end{aligned}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 4 & 5 & -3 & 0 \\ -1 & -3 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[\text{reduce}]{\text{row}} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

system has a free variable

$\therefore$ ~~xxxx~~ homogeneous system has a nontrivial solution

$\therefore$ vectors are L.D.