2.4

#12.

(a) $$A = \begin{bmatrix} 1 & -1 & -1 & 2 & 0 \\ -2 & 1 & 1 & -1 & 0 \\ 1 & 1 & -2 & 1 & 1 \end{bmatrix}$$

$$\Longrightarrow \operatorname{rref}(A) = \begin{array}{c} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -4/3 & 1/3 \\ 0 & 0 & 1 & -5/3 & -1/3 \end{bmatrix} \\ \begin{matrix} f & f & f & t & d \end{matrix} \end{array}$$

Let x_4 & x_5 be free, then

$$x_1 = x_4$$

$$x_2 = 4/3\, x_4 - 1/3\, x_5$$

$$x_3 = 5/3\, x_4 + 1/3\, x_5$$

So

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \operatorname{Ker}(A) \Longleftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_4 \\ 4/3\, x_4 - 1/3\, x_5 \\ 5/3\, x_4 + 1/3\, x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_4 \begin{bmatrix} 1 \\ 4/3 \\ 5/3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -1/3 \\ 1/3 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} 1 \\ 4/3 \\ 5/3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -1/3 \\ 1/3 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $NS(A)$.

(b) A basis for $RS(A)$ is

$\begin{bmatrix} 1 & 0 & 0 & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & -4/3 & 1/3 \end{bmatrix}$,

and $\begin{bmatrix} 0 & 0 & 1 & -5/3 & -1/3 \end{bmatrix}$

(c)

$$A^T = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & -2 \\ 2 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \operatorname{rref}(A^T) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

is a basis for $CS(A)$

i.e. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(d) $\operatorname{rank}(A) = \dim\left(CS(A)\right) = 3$.

2.5.

#3. $W\left(e^{2x}\cos x,\ e^{2x}\sin x\right)$

$$= \begin{vmatrix} e^{2x}\cos x & e^{2x}\sin x \\ 2e^{2x}\cos x - e^{2x}\sin x & 2e^{2x}\sin x + e^{2x}\cos x \end{vmatrix} \quad \cancel{2R_1 + R_2} \quad R_2 - 2R_1$$

$$= \begin{vmatrix} e^{2x}\cos x & e^{2x}\sin x \\ -e^{2x}\sin x & e^{2x}\cos x \end{vmatrix}$$

$$= \left(e^{2x}\right)^2 \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= e^{4x}\left(\cos^2 x + \sin^2 x\right)$$

$= e^{4x}$ nowhere zero

$\therefore\ e^{2x}\cos x,\ e^{2x}\sin x$ are L.I.