#8.

$$(TS)(ax+b)$$

$$= T(S(ax+b))$$

$$= T(ax - 2a + b)$$

$$= ax + 2(a) + (-2a + b)$$

$$= ax + b$$

#13. $L = D^4 + D^2$

$$\Longrightarrow \quad p(\lambda) = \lambda^4 + \lambda^2$$
$$= \lambda^2(\lambda^2+1)$$

roots : 0 (mult 2), $\pm i$ (mult 1).

$\lambda = 0$: gives the functions

$$y_1 = e^{0 \cdot x} = 1 \quad \text{and} \quad y_2 = x e^{0 \cdot x} = x$$

$\lambda = i$: gives the functions

$$y_3 = \cos x \quad \text{and} \quad y_4 = \sin x$$

<u>Claim:</u> y_1, y_2, y_3, y_4 are L. I.

<u>Pf:</u>

$$W(y_1, y_2, y_3, y_4) = \begin{vmatrix} 1 & x & \cos x & \sin x \\ 0 & 1 & -\sin x & \cos x \\ 0 & 0 & -\cos x & -\sin x \\ 0 & 0 & \sin x & -\cos x \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \\ 0 & \sin x & -\cos x \end{vmatrix}$$

$$= 1 \cdot 1 \cdot \begin{vmatrix} -\cos x & -\sin x \\ \sin x & -\cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x$$

$$= 1$$

$\neq 0$ everywhere

$\square$

Since $\dim(\ker(L)) = 4$ and $y_1, y_2, y_3, y_4 \in \ker(L)$ are L.I, then they form a basis.

#17. Let $g \in C^\infty(a,b)$

$T: C^\infty(a,b) \longrightarrow C^\infty(a,b)$

$T(f(x)) = g(x) f(x)$

Let $f_1, f_2 \in C^\infty(a,b)$, then

$$T\left(f_1(x) + f_2(x)\right) = T\left((f_1 + f_2)(x)\right)$$

$$= g(x) \cdot (f_1 + f_2)(x)$$

$$= g(x) \cdot \left(f_1(x) + f_2(x)\right)$$

$$= g(x) f_1(x) + g(x) f_2(x)$$

$$= T(f_1(x)) + T(f_2(x))$$

$$
\begin{aligned}
T(c \cdot f(x)) &= T((c \cdot f)(x)) \\
&= g(x) \cdot (cf)(x) \\
&= g(x) \cdot (c \cdot f(x)) \\
&= c \cdot (g(x) \cdot f(x)) \\
&= c \cdot T(f(x))
\end{aligned}
$$