5.6

#6.

$$V = \{ y : y'' + y = 0 \}$$

$$= \operatorname{Ker}(D^2 + 1)$$

Find a basis α of V.

Characteristic poly of $y'' + y = 0$ is

$$p(\lambda) = \lambda^2 + 1 = 0$$

roots $\lambda = \pm i$

pick $\lambda = i = 0 + 1 \cdot i$

Let

$$y_1 = e^{0 \cdot x} \cdot \cos(1 \cdot x) = \cos x$$

$$y_2 = e^{0 \cdot x} \cdot \sin(1 \cdot x) = \sin x$$

Claim: y_1 & y_2 are a basis of V.

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$= \cos^2 x + \sin^2 x = 1 \neq 0$$

for all x.

$\Rightarrow y_1$ & y_2 are L.I.

Since $\dim(V) = 2$, then $\alpha = \{y_1, y_2\}$

is a basis of V.

L

Find the matrix of $D^2 : V \to V$ w.r.t. α.

$$D^2 \cos x = -\cos x$$

$$D^2 \sin x = -\sin x$$

$$[D^2]_\alpha^\alpha = \left[[D^2 \cos x]_\alpha \;\; [D^2 \sin x]_\alpha \right]$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$$

(a) The eigenvalue of $[D^2]_\alpha^\alpha$ is $\lambda = -1$,

so the eigenvalue of D^2 is $\lambda = -1$.

A basis for the eigenspace E_{-1} of

$[D^2]_\alpha^\alpha$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Longrightarrow \varphi_\alpha^{-1}\begin{bmatrix}1\\0\end{bmatrix} = \cos x$ & $\varphi_\alpha^{-1}\begin{bmatrix}0\\1\end{bmatrix} = \sin x$

is a basis for the eigenspace V_{-1}

of D^2.

(b) D^2 is diagonalizable since $[D^2]_\alpha^\alpha$ is

(it's already diagonal).

(c) The Jordan canonical form of D^2

is itself since the Jordan

canonical form of $[D^2]_\alpha^\alpha$ is itself.

(it's diagonal).

6.2

#18. $Y' = \underbrace{\begin{bmatrix} 3 & 1 & -1 \\ 0 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}}_{A} Y, \quad Y(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$p(\lambda) = \begin{vmatrix} \lambda - 3 & -1 & 1 \\ 0 & \lambda & 2 \\ 0 & -1 & \lambda - 2 \end{vmatrix}$$

$$= (\lambda - 3)\left[\lambda(\lambda - 2) + 2\right]$$

$$= (\lambda - 3)(\lambda^2 - 2\lambda + 2)$$

$$= (\lambda - 3)(\lambda - (1+i))(\lambda - (1-i))$$

$$E_3: \left[\begin{array}{ccc|c} 0 & -1 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right] \longrightarrow \begin{array}{c} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{ccc} f & d & d \end{array} \end{array}$$

x is free, $y = z = 0$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_3 \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} = x \cdot e_1$$

$$E_{1+i} : \left[\begin{array}{ccc|c} -2+i & -1 & 1 & 0 \\ 0 & 1+i & -2 & 0 \\ 0 & -1 & -1+i & 0 \end{array}\right]$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1-i \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} \quad d & d & f \end{matrix}$$

z free, $x = z$ & $y = -(1-i)z$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in E_{1+i} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ (i-1)z \\ z \end{bmatrix} = z \begin{bmatrix} 1 \\ i-1 \\ 1 \end{bmatrix}$$

basis of E_{1+i}

$$\Longrightarrow \overline{\begin{bmatrix} 1 \\ i-1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 1 \\ -1-i \\ 1 \end{bmatrix} \text{ is a basis of } E_{1-i}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1+i & -1-i \\ 0 & 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1+i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$$

The solution $Z_1 = \begin{bmatrix} e^{3x} \\ 0 \\ 0 \end{bmatrix}$ gives the

solution

$$Y_1 = PZ_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1+i & -1-i \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3x} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{3x} \\ 0 \\ 0 \end{bmatrix}$$

The complex-valued solution $Z_2 = \begin{bmatrix} 0 \\ e^{(1+i)x} \\ 0 \end{bmatrix}$

will give two real-valued solutions.

$$Z_2 = \begin{bmatrix} 0 \\ e^{(1+i)x} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ e^x \cos x + i \cdot e^x \sin x \\ 0 \end{bmatrix}$$

$$\Longrightarrow PZ_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1+i & -1-i \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ e^x \cos x + i \cdot e^x \sin x \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^x \cos x + i \cdot e^x \sin x \\ -e^x(\cos x + \sin x) + i e^x (\cos x - \sin x) \\ e^x \cos x + i e^x \sin x \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^x \cos x \\ -e^x(\cos x + \sin x) \\ e^x \cos x \end{bmatrix}}_{Y_2} + i \underbrace{\begin{bmatrix} e^x \sin x \\ e^x(\cos x - \sin x) \\ e^x \sin x \end{bmatrix}}_{Y_3}$$

The vectors Y_1, Y_2, Y_3 are solutions to $Y' = AY$

Since the dimension of the space of solutions to $Y' = AY$ is 3 it suffices to show Y_1, Y_2, Y_3 are L.I to see they form a basis. Compute the Wronskian

$$W(Y_1, Y_2, Y_3) = \begin{vmatrix} e^{3x} & e^x \cos x & e^x \sin x \\ 0 & -e^x(\cos x + \sin x) & e^x(\cos x - \sin x) \\ 0 & e^x \cos x & e^x \sin x \end{vmatrix}$$

$$= e^{3x} \cdot \begin{vmatrix} -e^x(\cos x + \sin x) & e^x(\cos x - \sin x) \\ e^x \cos x & e^x \sin x \end{vmatrix}$$

$$= e^{3x}\cdot\left(e^{x}\right)^{2}\cdot\begin{vmatrix} -(\cos x + \sin x) & \cos x - \sin x \\ \cos x & \sin x \end{vmatrix}\begin{array}{l} R_1 + R_2 \\ R_2 \end{array}$$

$$= e^{5x}\cdot\begin{vmatrix} -\sin x & \cos x \\ \cos x & \sin x \end{vmatrix}$$

$$= e^{5x}\left(-\sin^2 x - \cos^2 x\right)$$

$$= -e^{5x}$$

$\neq 0$ for all x.

$\Longrightarrow Y_1, Y_2, Y_3$ are L. I.

General solution to $Y' = AY$ is

$$Y_H = \begin{bmatrix} c_1 e^{3x} + c_2 e^{x}\cos x + c_3 e^{x}\sin x \\ -c_2 e^{x}(\cos x + \sin x) + c_3 e^{x}(\cos x - \sin x) \\ c_2 e^{x}\cos x + c_3 e^{x}\sin x \end{bmatrix}$$

$$Y(0) = \begin{bmatrix} c_1 + c_2 \\ -c_2 + c_3 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\Longrightarrow \quad c_2 = -1$$

$$\Longrightarrow \quad c_3 = -1$$

$$\Longrightarrow \quad c_1 = 2$$

The solution to the IVP is

$$Y = \begin{bmatrix} 2e^{3x} - e^x \cos x - e^x \sin x \\ e^x(\cos x + \sin x) - e^x(\cos x - \sin x) \\ -e^x \cos x - e^x \sin x \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{3x} - e^x \cos x - e^x \sin x \\ 2e^x \sin x \\ -e^x \cos x - e^x \sin x \end{bmatrix}$$