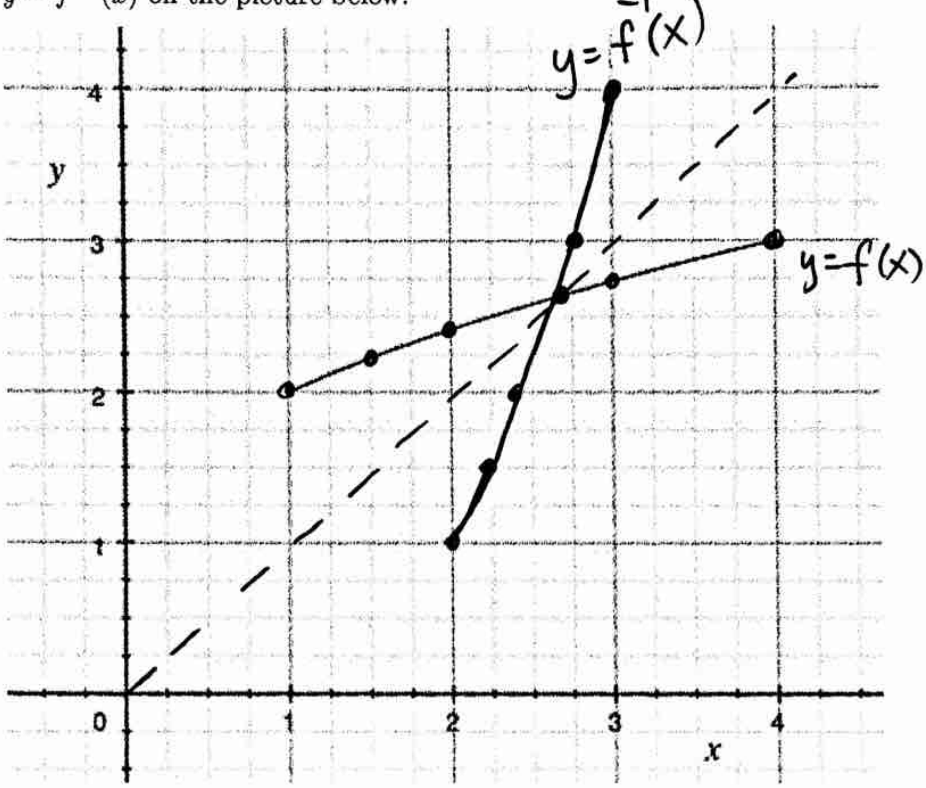
Math 242 Final, Fall 2015

Name: Solutions by D. Yuen

Question	Points	Score
1	3	
2	4	
3	12	
4	12	
5	30	
6	12	
7	32	
8	10	
9	6	
10	7	
11	6	
12	16	
Total:	150	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. (3 points) Let $f(x)$ be a function with domain $[1,4]$ and graph as pictured. Sketch the graph $y = f^{-1}(x)$ on the picture below.



2. (4 points) Say $g(x) = 2x - 3$. Find, and simplify, a formula for $g^{-1}(x)$.

Set $x = 2y - 3$ solve for y

$$x + 3 = 2y$$

$$\frac{x+3}{2} = y$$

$$g^{-1}(x) = \frac{x+3}{2}$$

OR

$$g^{-1}(x) = \frac{1}{2}x + \frac{3}{2}$$

3. Compute the derivatives of the following functions. You do not have to simplify your answers.

(a) (6 points) $f(x) = \ln(2x)2^x$.

product rule

$$f'(x) = \frac{2}{2x} \cdot 2^x + \ln(2x) \cdot 2^x \ln 2$$

(b) (6 points) $f(x) = x^{\sin x}$.

Logarithmic differentiation.

$$y = x^{\sin x}$$

$$\ln y = (\sin x)(\ln x)$$

$$\frac{d}{dx} \quad \frac{y'}{y} = (\cos x)(\ln x) + (\sin x)\frac{1}{x}$$

$$y' = y\left((\cos x)(\ln x) + \frac{(\sin x)}{x}\right)$$

$$f'(x) = x^{\sin x}\left((\cos x)\ln x + \frac{\sin x}{x}\right)$$

4. Compute the following limits. You must justify your solution using algebraic manipulations and / or l'Hôpital's rule for full credit.

(a) (6 points) $\lim_{n\to\infty} \frac{n^3}{e^n}$.

Type $\frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{n\to\infty} \frac{3n^2}{e^n}$ Type $\frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{n\to\infty} \frac{6n}{e^n}$ Type $\frac{\infty}{\infty}$

$\overset{L'H}{=} \lim_{n\to\infty} \frac{6}{e^n}$ Type $\frac{6}{\infty}$

$= 0$

(b) (6 points) $\lim_{x\to 0^+} (1+x)^{2/x}$.

Type 1^∞

Set $L=$

$\ln L = \lim_{x\to 0^+} \frac{2}{x} \ln(1+x)$

$= \lim_{x\to 0^+} \frac{2\ln(1+x)}{x}$ Type $\frac{0}{0}$

$\overset{L'H}{=} \lim_{x\to 0^+} \frac{2\,\frac{1}{1+x}}{1}$ Type $\frac{2\cdot 1}{1}$

$\ln L = 2$

Hence $L = e^2$

5. Compute the following integrals, or say if they diverge.

(a) (10 points) $\int \sin^2(x)\cos^5(x)dx$.

ODD power of cos

Let $u = \sin(x)$

$du = \cos x \, dx$

$= \int \sin^2 x \cos^4 x \cos x \, dx$

$= \int \sin^2 x \, (1-\sin^2 x)^2 \cos x \, dx$

because $\cos^2 x = 1 - \sin^2 x$ from formula page in back.

$= \int u^2 (1-u^2)^2 du$

$= \int u^2 (1 - 2u^2 + u^4) du$

$= \int (u^2 - 2u^4 + u^6) du$

$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$

$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$

(b) (10 points) $\int \frac{1}{\sqrt{x^2+9}} dx.$

Pattern $\sqrt{\square^2 + a^2}$

Let $x = 3\tan\theta$

So that

$\sqrt{x^2 + 3^2} = 3\sec\theta$

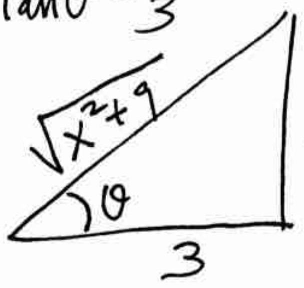
$dx = 3\sec^2\theta \, d\theta$

$$= \int \frac{3\sec^2\theta \, d\theta}{3\sec\theta}$$

$$= \int \sec\theta \, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$\tan\theta = \frac{x}{3}$



So $\sec\theta = \frac{\sqrt{x^2+9}}{3}$

$$= \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + C$$

Note: It is possible, but not necessary, to simplify this further.

(c) (10 points) $\int_3^\infty \frac{1}{2x^2+3x+1}dx$.

Check discriminant

$B^2-4AC = 3^2-4\cdot2\cdot1 = 1$

> 0

so it can be factored.

Do indefinite integral first

$$\frac{1}{2x^2+3x+1} = \frac{1}{(2x+1)(x+1)}$$

$$= \frac{A}{2x+1} + \frac{B}{x+1}$$

Clear denominators

$$1 = A(x+1) + B(2x+1)$$

Set $x=-1 \Rightarrow 1 = 0 + B(-1) \Rightarrow -1 = B$

Set $x = -\frac{1}{2} \Rightarrow 1 = A(\frac{1}{2}) + 0 \Rightarrow 2 = A$

Get $\int \left(\frac{2}{2x+1} + \frac{-1}{x+1}\right)dx = 2\,\frac{\ln|2x+1|}{2} - \ln|x+1|$

$$= \ln|2x+1| - \ln|x+1|$$

$$\int_3^\infty \frac{dx}{2x^2+3x+1} = \lim_{b\to\infty}\int_3^b \frac{dx}{2x^2+3x+1} = \lim_{b\to\infty}\left(\ln|2x+1| - \ln|x+1|\right)\Big|_3^b$$

$$= \lim_{b\to\infty}\left(\ln|2b+1| - \ln|b+1| - (\ln 7 - \ln 4)\right)$$

$$= \lim_{b\to\infty} \ln\left|\frac{2b+1}{b+1}\right| - \ln 7 + \ln 4$$

$\cdot\, \frac{1/b}{1/b}$

$$= \lim_{b\to\infty} \ln\left|\frac{2+\frac{1}{b}}{1+\frac{1}{b}}\right| - \ln 7 + \ln 4$$

$$= \ln\left|\frac{2+0}{1+0}\right| - \ln 7 + \ln 4 = \ln 2 - \ln 7 + \ln 4$$

6. Compute the sum of each of the convergent series below. Simplify your answers.

(a) (5 points) $\sum_{n=0}^{\infty} \frac{1-2^n}{3^n}$.

Two geometric series

$$\sum_{n=\infty}^{\infty} \frac{1}{3^n} - \sum_{n=0}^{\infty} \frac{2^n}{3^n}$$

$$= \frac{\frac{1}{3^0}}{1-\frac{1}{3}} - \frac{\left(\frac{2}{3}\right)^0}{1-\frac{2}{3}} = \frac{1}{\frac{2}{3}} - \frac{1}{\frac{1}{3}} = \frac{3}{2} - 3 = -\frac{3}{2}$$

(b) (7 points) $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$.

Better be telescopic

$$\frac{1}{n(n-1)} = \frac{A}{n} + \frac{B}{n-1}$$

$$1 = A(n-1) + Bn$$

Set $n=1 \Rightarrow 1 = 0 + B \Rightarrow 1 = B$

Set $n=0 \Rightarrow 1 = A(-1) + 0 \Rightarrow -1 = A$

$$= \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= \sum_{n=2}^{\infty} \left(g(n) - g(n+1)\right) \text{ where } g(n) = \frac{1}{n-1}.$$

$$= g(2) - \lim_{n\to\infty} g(n)$$

$$= \frac{1}{2-1} - \lim_{n\to\infty} \frac{1}{n-1}$$

$$= \frac{1}{1} - 0 = 1.$$

7. For each of the following series, say whether they converge or diverge. For full credit, you must justify your solutions, and state clearly which test(s) you are using (if any).

(a) (8 points) $\sum_{n=1}^{\infty} \frac{-4}{n^{3/2}}$.

Always negative, so multiply by -1.

Convergent if and only if the following is convergent

$$\sum_{n=1}^{\infty} \frac{4}{n^{3/2}}$$

Just a p-series with $p = \frac{3}{2} > 1$. Hence convergent.

(b) (8 points) $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+n}$.

Alternating series with $u_n = \frac{\sqrt{n}}{1+n}$ (Note $u_n > 0$).

- $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{\sqrt{n}}{1+n} \frac{\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}} + \sqrt{n}} = 0$

because $\frac{1}{\sqrt{n}} + \sqrt{n} \to \infty$

- Decreasing? Not obvious so check derivative: $\frac{d}{dn}\left(\frac{\sqrt{n}}{1+n}\right) = \frac{\frac{1}{2}n^{-1/2}(1+n) - n^{1/2}(1)}{(1+n)^2}$

$$= \frac{\frac{1}{2}n^{-1/2} + \frac{1}{2}n^{1/2} - n^{1/2}}{(1+n)^2} = \frac{\frac{1}{2}n^{-1/2} - \frac{1}{2}n^{1/2}}{(1+n)^2} = \frac{\frac{1}{2}n^{-1/2}(1-n)}{(1+n)^2}$$

This is ≤ 0 for $n \geq 1$

Thus $\{u_n\}$ is decreasing. By Alternating series Test, this series converges.

(c) (8 points) $\sum_{n=0}^{\infty} \frac{2^n n^3}{(n+2)!}$. ← See factorial. Try Ratio Test, positive series yes.

$$\rho = \lim_{n\to\infty} \frac{2^{n+1}(n+1)^3}{(n+3)!} \Big/ \frac{2^n n^3}{(n+2)!}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}(n+1)^3}{(n+3)!} \frac{(n+2)!}{2^n n^3} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \frac{(n+1)^3}{n^3} \frac{(n+2)!}{(n+3)!}$$

$$= \lim_{n\to\infty} \frac{2}{1} \left(1+\frac{1}{n}\right)^3 \frac{1}{n+3}$$

$$= 2 \cdot 1^3 \cdot 0 = 0. \Leftarrow \rho < 1$$

So series converges.

(d) (8 points) $\sum_{n=1}^{\infty} \frac{\sin n}{n+n^2}$. ← sin(n) is +/- not in an alternating way.

Only hope is absolute convergence.

Consider $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n+n^2}\right|$

$$\left|\frac{\sin n}{n+n^2}\right| \leq \frac{1}{0+n^2}.$$

Since $\sum \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$) then $\sum_{n=1}^{\infty} \left|\frac{\sin n}{n+n^2}\right|$ converges.

Hence $\sum_{n=1}^{\infty} \frac{\sin n}{n+n^2}$ converges absolutely.

8. (10 points) Find the values of x for which the power series

$$\sum_{n=0}^{\infty} \frac{1}{n2^n}(x+3)^n$$

(a) converges absolutely; (b) converges conditionally; (c) diverges. Justify your answers.

Absolute Root test: OR Absolute Ratio Test:

$$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{n2^n}(x+3)^n\right|} = \lim_{n\to\infty} \frac{|x+3|}{\sqrt[n]{n}\ 2} = \frac{|x+3|}{2}$$

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)2^{n+1}}(x+3)^{n+1}}{\frac{1}{n2^n}(x+3)^n} \right| = \lim_{n\to\infty} \frac{n}{n+1}\,\frac{1}{2}\,|x+3| = \frac{|x+3|}{2}$$

Solve $\rho < 1 \Rightarrow \frac{|x+3|}{2} < 1 \Rightarrow |x+3| < 2$ converges absolutely

$$-3-2 < x < -3+2$$

$$-5 < x < -1$$

Endpoints are $x = -1 \Rightarrow \sum \frac{1}{n2^n} 2^n = \sum \frac{1}{n}$ diverges (p-series $p=1$)

$x = -5 \Rightarrow \sum \frac{1}{n2^n}(-2)^n = \sum \frac{(-1)^n}{n}$ converges conditionally

because converges by Alternating series Test: $u_n = \frac{1}{n}$ decreasing and $\lim_{n\to\infty} \frac{1}{n} = 0$.

But $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ diverges.

Summary:

Converges absolutely for $-5 < x < -1$

Converges conditionally for $x = -5$

Diverges otherwise (namely for $x \geq -1$ and for $x < -5$)

9. (6 points) Find the order two Taylor polynomial for $\arctan(x)$, centered at $a = 1$.

Call $f(x)$ " $\quad f(1) = \arctan 1 = \frac{\pi}{4}$

$f'(x) = \frac{1}{x^2+1} \qquad f'(1) = \frac{1}{2}$

$f''(x) = \frac{-1}{(x^2+1)^2} \cdot 2x \qquad f''(1) = \frac{-2}{2^2} = -\frac{1}{2}$

$$P_2(x) = \frac{\pi}{4} + \frac{1}{2}(x-1) + \frac{1}{2!}\left(-\frac{1}{2}\right)(x-1)^2$$
$$= \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2$$

10. (7 points) Let $f(x) = \cos(4x)$, and say it is replaced by $P_2(x) = 1 - 16x^2/2$ (its order 2 Taylor polynomial, centered at 0). Use a technique from the course to estimate the error $R_2(x)$ when $|x| < 1/2$.

Need $M_3 \geq \max$ ~~of~~ $f'''(x)$ on $|x| < \frac{1}{2}$.

$f'(x) = -4\sin(4x)$

$f''(x) = -16\cos(4x)$

$f'''(x) = +64\sin(4x)$. Thus $|f'''(x)| \leq 64$.

We may use $M_3 = 64$.

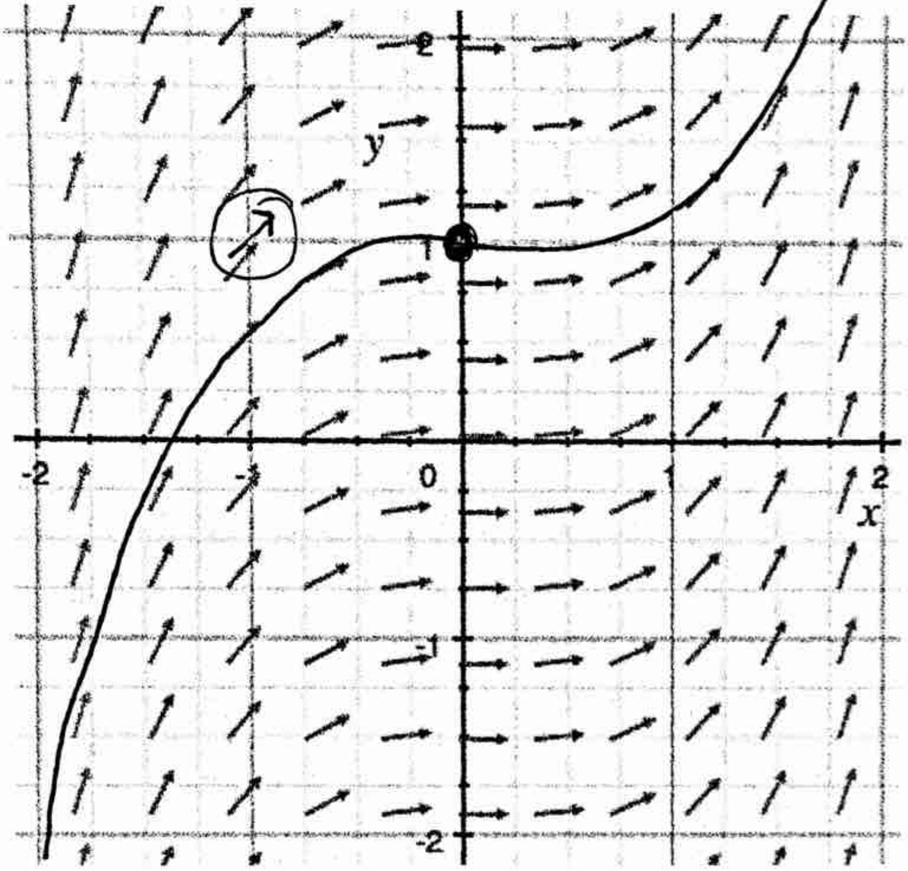
Then using formula for Taylor Remainder from formula page,

$$|R_2(x)| \leq \frac{M_3|x-0|^3}{3!} \leq \frac{64\left(\frac{1}{2}\right)^3}{3!} = \frac{64\,\frac{1}{8}}{6} = \frac{4}{3},$$

for $|x| < \frac{1}{2}$

(b)

11. Consider the slope field pictured below.



(a) (3 points) Which of the differential equations below matches this slope field?

(a) $\dfrac{dy}{dx} = \dfrac{x}{y}$, (b) $\dfrac{dy}{dx} = x^2$, (c) $\dfrac{dy}{dx} = xy$, (d) $\dfrac{dy}{dx} = y^2$.

	(a)	(b)	(c)	(d)
(slope at (0,1)) = 0	0	0	0	1
(slope at (−1,1)) = 1	−1	1	−1	1

(b) (3 points) Sketch the solution to this differential equation that satisfies $y(0) = 1$ on the slope field.

12. Solve the following differential equations. Either give the general solution, or solve for a particular solution satisfying the given initial conditions. Your solution must give an explicit formula for y for full credit.

(a) (8 points) $\frac{dy}{dx} = y^2 + 1.$

separable

$$\frac{dy}{y^2+1} = dx$$

$$\int \frac{dy}{y^2+1} = \int dx$$

$$\arctan(y) = x + C$$

$$y = \tan(x + C)$$

(b) (8 points) $y' + y = 2x, \quad y(0) = 1.$ Linear

$P = 1$

$\int P\,dx = \int 1\,dx = x$ for integrating factor

$v = e^{\int P\,dx} = e^x.$

$e^x y' + e^x y = 2xe^x$

$\frac{d}{dx}(e^x y) = 2xe^x$

$e^x y = \int 2x e^x\,dx$ integration by parts

$u = 2x \quad dv = e^x dx$

$du = 2dx \quad v = e^x$

$= 2xe^x - \int 2e^x\,dx$

$e^x y = 2xe^x - 2e^x + C$

$y = 2x - 2 + Ce^{-x}.$

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Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{(true for } -1 < x < 1\text{)}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \quad \text{(true for all } x\text{)}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \quad \text{(true for } x < -1 \text{ and } x > 1\text{)}$$

- Pythagorean identities (true for all x where the functions involved are defined).

$$\sin^2(x) + \cos^2(x) = 1, \quad \tan^2(x) + 1 = \sec^2(x), \quad 1 + \cot^2(x) = \csc^2(x).$$

- Reduction of power formulas / double angle formulas for sine and cosine (true for all x).

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x)), \quad \sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

- Addition formulas for sine and cosine (true for all x and y).

$$\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$

$$\sin(x)\cos(y) = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

- Integrals of tangent and secant.

$$\int \tan(x)dx = -\ln|\cos(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C.$$

- Standard power series expansions (centered at $a = 0$).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{(valid for all } x\text{)}.$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{(valid for all } x\text{)}.$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{(valid for all } x\text{)}.$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad \text{(valid for } |x| < 1\text{)}.$$

$$(1+x)^m = \sum_{n=0}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n \qquad \text{(valid for } |x| < 1\text{)}.$$

- Error estimate for approximations by Taylor polynomials.
  Say $f(x)$ is a function with derivatives of all orders on an interval $[b, c]$, and a is a point in $[b, c]$. Say $P_N(x)$ is the N^{th} Taylor polynomial for $f(x)$ centered at a, and $R_N(x) = f(x) - P_N(x)$ is the error when approximating $f(x)$ by $P_N(x)$. Then for all x in $[b, c]$

$$|R_N(x)| \leq \frac{M_{N+1}|x-a|^{N+1}}{(N+1)!},$$

where M_{N+1} is the largest value taken by the $(N+1)^{\text{st}}$ derivative of $f(x)$ on $[b, c]$.