Math 242 Final

Name: ______________________________

Please circle your section:

Recitation 1	Thurs 12-12:50	TA - Dan Flores
Recitation 2	Thurs 1:30-2:20	TA - Dan Flores
Recitation 3	Tues 9-9:50	TA - Vince Chung
Recitation 4	Tues 12-12:50	TA - Vince Chung
Recitation 5	Wed 9:30-10:20	TA - Lance Ferrer
Recitation 6	Wed 12:30-1:20	TA - Lance Ferrer
Recitation 7	Fri 10:30-11:20	TA - Ikenna Nometa
Recitation 8	Fri 12:30-1:20	TA - Ikenna Nometa
Recitation 9	Fri 9:30-10:20	TA - Dan Flores

Question	Points	Score
1	12	
2	10	
3	12	
4	10	
5	13	
6	12	
7	15	
8	15	
9	13	
10	6	
11	17	
12	10	
13	5	
Total:	150	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. Find the derivative of each of the following functions.

   (a) (5 points) $f(x) = 2\sin^{-1}(x^2)$

   (b) (7 points) $g(x) = 2^{x+\ln(x)}$

2. (10 points) Evaluate the following integral: $\int x \sec^2 x \, dx$

3. Evaluate the following integrals.

(a) (7 points) $\displaystyle\int_0^{\pi/6} \cos^3(3x)\sin^2(3x)\,dx$

(b) (5 points) $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

4. (10 points) Evaluate the following integral. (Hint: Use trigonometric substitution.)

$$\int \frac{dx}{(9 - x^2)^{3/2}}$$

5. Determine whether the following improper integrals converge or diverge, and evaluate those that converge.

(a) (5 points) $\displaystyle\int_1^3 \frac{2}{\sqrt{x-1}}\,dx$

(b) (5 points) $\displaystyle\int_3^\infty \frac{2}{\sqrt{x-1}}\,dx$

(c) (3 points) What does your answer to (b) tell you about the series $\displaystyle\sum_{n=3}^{\infty} \frac{2}{\sqrt{n-1}}$?

6. Evaluate the following limits. If a limit does not exist write DOES NOT EXIST.

(a) (4 points) $\lim_{n\to\infty} (-1)^n \frac{n}{n+1}$

(b) (4 points) $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{3}\right)^k$

(c) (4 points) $\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n$

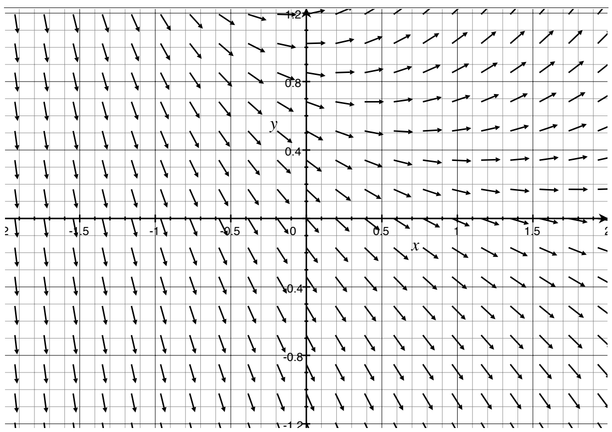
7. For each of the following series decide if it converges or diverges and explain why.

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+1}$

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n \cdot n!}$

(c) (5 points) $\displaystyle\sum_{n=3}^{\infty} (-1)^n \frac{\ln(n)}{n}$

8. Consider the differential equation $y' = y - e^{-x}$, with slope field pictured below.



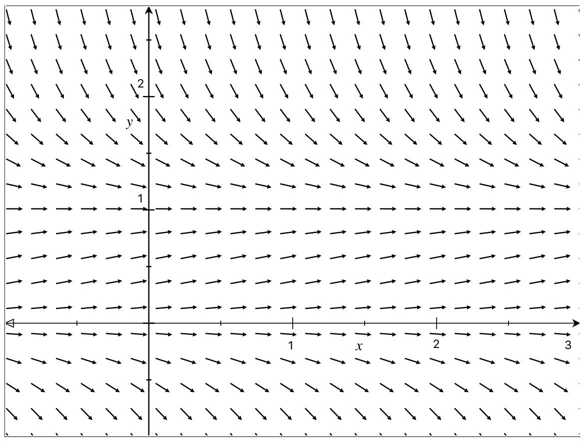
(a) (4 points) Sketch (on the slope field) the solutions satisfying $y(0) = 0$ and $y(0) = 1$.

(b) (7 points) Find the general solution of the differential equation.

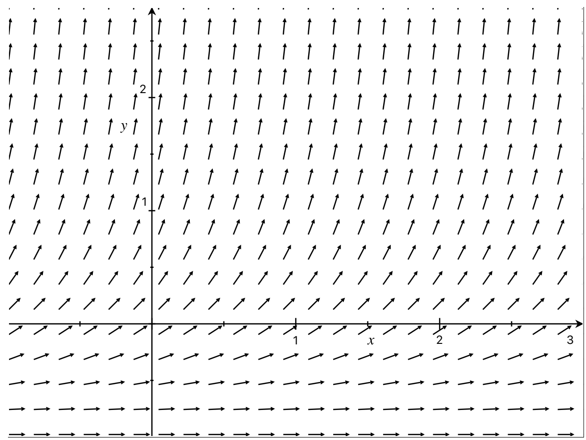
(c) (4 points) Find the particular solution satisfying $y(0) = 0$.

9. Consider the differential equation: $y' = y(1-y)$.

(a) (4 points) Which of the following plots represents the direction field of this differential equation? Circle your answer.



(I)



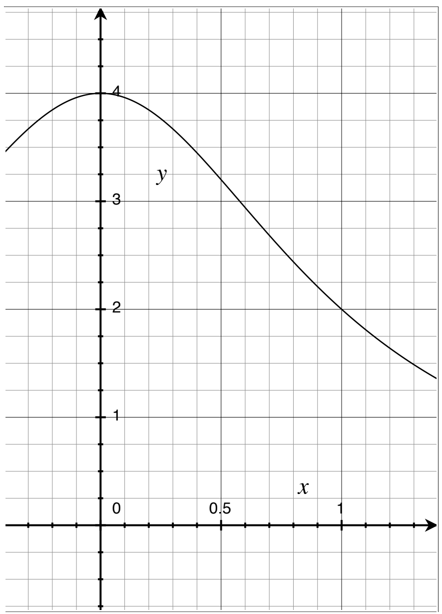
(II)

(b) (2 points) If y is the solution satisfying $y(0) = 2$, what is $\lim_{x\to\infty} y$?
(Hint: You can read this directly off the slope field.)

(c) (7 points) Solve the differential equation.

10. In this problem, you will use numerical integration to estimate $\pi = \int_0^1 \frac{4\,dx}{1+x^2}$.

(a) (2 points) The graph the function $y = 4/(1+x^2)$ between $x = 0$ and $x = 1$ is shown below. On the graph draw, and shade in, the trapezoids whose area is computed by the Trapezoidal Rule with $n = 2$.



(b) (4 points) Use the Trapezoidal Rule with $n = 2$ to estimate the integral $\int_0^1 \frac{4\,dx}{1+x^2}$.

Your answer should be a fraction (or decimal number).

11. Consider the power series: $\sum_{n=1}^{\infty} \frac{(-1)^n (x-2)^n}{n \cdot 2^n}$

(a) (13 points) Find its interval of convergence. (Hint: Check the endpoints.)

(b) (2 points) What is its radius of convergence?

(c) (2 points) For which values of x does the series converge absolutely?

12. Consider the function $f(x) = x\cos(3x)$.

   (a) (5 points) Write down the Taylor series for $f(x)$ based at $a = 0$. (Hint: Manipulate a 'famous Maclaurin series' – do not calculate derivatives.)

   (b) (5 points) Use your result in (a) to find $f^{(5)}(0)$ (the fifth derivative of f).

13. (5 points) The degree 3 Taylor polynomial centered at $a = 0$ for the function $\sin(x/2)$ is

$$\sin(x/2) \approx \frac{x}{2} - \frac{x^3}{8 \cdot 3!}. \tag{1}$$

Estimate the error in this approximation when $|x| < 0.1$.

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$$

- Trigonometric identities.

$$\sin^2 x + \cos^2 x = 1 \qquad \sin(x+y) = \sin x\cos y + \cos x\sin y$$
$$1 + \tan^2 x = \sec^2 x \qquad \cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$1 + \cot^2 x = \csc^2 x \qquad \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$$
$$\sin^2 x = \frac{1}{2}(1-\cos(2x))$$
$$\cos^2 x = \frac{1}{2}(1+\cos(2x))$$
$$\sin x\cos x = \frac{1}{2}\sin(2x)$$
$$\sin x\sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$
$$\cos x\cos y = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$
$$\sin x\cos y = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

- Integrals of trigonometric functions.

$$\int \tan x \; dx = \ln|\sec x| + C$$
$$\int \cot x \; dx = \ln|\sin x| + C$$
$$\int \sec x \; dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x \; dx = -\ln|\csc x + \cot x| + C$$

- Trapezoidal Rule and Simpson's Rule.

$$T = \frac{\Delta x}{2}\Big(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\Big)$$
$$S = \frac{\Delta x}{3}\Big(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\Big)$$

- Error estimates for Trapezoidal Rule and Simpson's Rule.

$$|E_T| \le \frac{M(b-a)^3}{12n^2}, \quad \text{where } |f''(x)| \le M \text{ for all } x \text{ in } [a,b]$$
$$|E_S| \le \frac{M(b-a)^5}{180n^4}, \quad \text{where } |f^{(4)}(x)| \le M \text{ for all } x \text{ in } [a,b]$$

- Famous Maclaurin series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (R = \infty)$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (R = \infty)$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (R = \infty)$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad (R = 1)$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad (R = 1)$$

- Error estimate for approximations by Taylor polynomials.

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(t)| \le M$ for all t between a and x.