Math 242 Final

Name: ______________________________

Section:_____

If you do not know your section, fill the following:

Instructor:______________________________

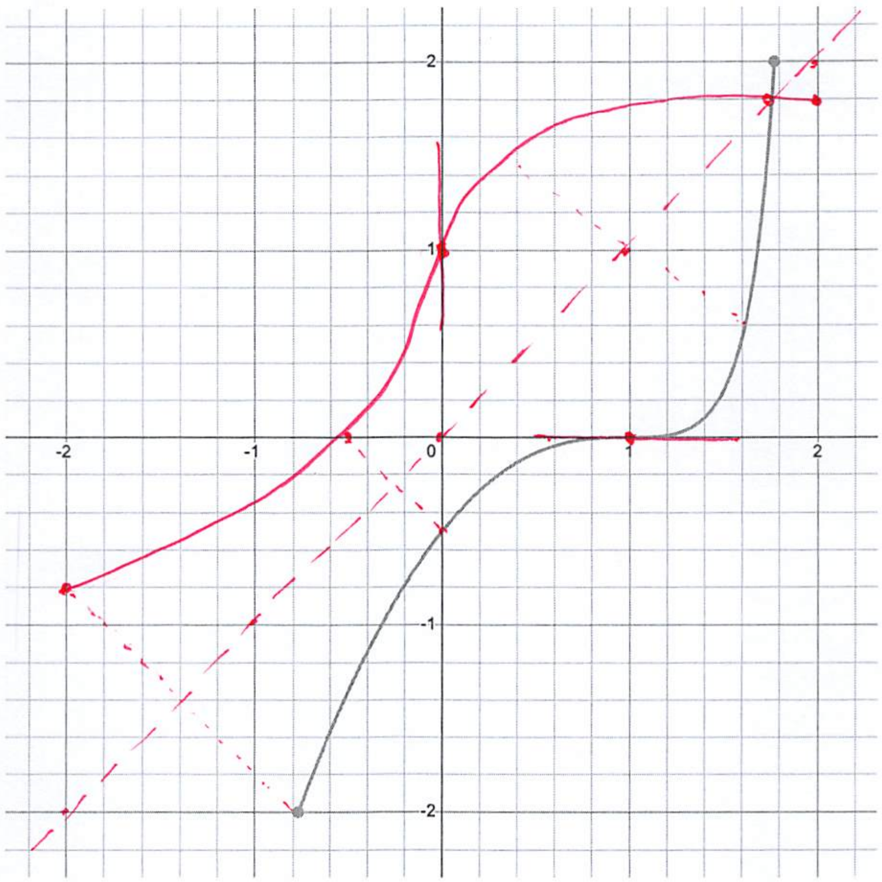
TA:______________________________

Solutions by Kenny

Question	Points	Score
1	12	
2	4	
3	4	
4	12	
5	32	
6	12	
7	28	
8	10	
9	7	
10	7	
11	6	
12	16	
Total:	150	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. The graph of a function $f(x)$ is sketched on the picture below.



(a) (4 points) Sketch the graph of $y = f^{-1}(x)$ on the same plot.

(b) (4 points) Specify the domain of the function $y = f^{-1}(x)$.

$[-2, 2]$

(c) (4 points) List all values of x within the interval $(-2, 2)$ where the derivative $y'(x) = \frac{d}{dx} f^{-1}(x)$ does not exist. (If the derivative exists at every point on the interval, state that.)

$x = 0$

2. (4 points) Let $f(x) = \sin(3x)$ for $-\pi/6 \le x \le \pi/6$.
Find the value of the derivative of the inverse function $(f^{-1})'$ at $-\sqrt{3}/2$

$(f^{-1})'\left(-\frac{\sqrt{3}}{2}\right)$.

$$f^{-1}(-\sqrt{3}/2) = x \iff f(x) = \sin(3x) = -\frac{\sqrt{3}}{2}$$

$$3x = \arcsin(-\sqrt{3}/2) = -\frac{\pi}{3}$$

$$f'(x) = 3\cos(3x)$$

$$x = -\frac{\pi}{9}$$

$$(f^{-1})'\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{f'(f^{-1}(-\sqrt{3}/2))}$$

$$= \frac{1}{f'(-\pi/9)} = \frac{1}{3\cos(3\cdot(-\pi/9))} = \frac{1}{3\cos(-\pi/3)}$$

$$= \frac{1}{3\cdot 1/2} = 2/3$$

3. (4 points) Compute the derivative of the function

$y = (\sin x)^{\sin x}$.

$$\ln y = \sin x \cdot \ln(\sin x)$$

$$\frac{y'}{y} = \left(\cos x \cdot \ln(\sin x) + \sin x \cdot \frac{1}{\sin x} \cdot \cos x\right)$$

$$y' = (\sin x)^{\sin x}\left(\cos x \cdot \ln(\sin x) + \cos x\right)$$

4. Compute the following limits. You must justify your solution using algebraic manipulations and/or l'Hôpital's rule for full credit.

(a) (6 points) $\lim_{x\to 0^+} (\tan x)^{(1/x^2)}$

Type 0^∞

not an indeterminate power.

limit is just 0.

(b) (6 points) $\lim_{x\to 0^+} (1+\sin x)^{\cot x}$.

Type 1^∞

Set $L = \lim_{x\to 0^+} (1+\sin x)^{\cot x}$

$\Rightarrow \ln L = \lim_{x\to 0^+} \cot x \cdot \ln(1+\sin x)$ Type $\infty \cdot 0$ use $ab = \frac{a}{1/b}$

$= \lim_{x\to 0^+} \frac{\ln(1+\sin x)}{\tan x}$ Type $\frac{0}{0}$

$\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+\sin x} \cdot \cos x}{\sec^2 x}$

$= \lim_{x\to 0^+} \frac{\cos^3 x}{1+\sin x}$

$= \frac{\cos^3(0)}{1+\sin(0)} = \frac{1}{1+0} = 1$

$\therefore L = e^1 = e$

5. Compute the following integrals, or say if they diverge.

For full credit, make sure that you present a clear justification which supports your claim for every divergent integral.

(a) (8 points) $\int_2^{10} \frac{dx}{x^2 - 8x + 7}$ $= \underbrace{\int_2^{10} \frac{dx}{(x-1)(x-7)}}_{\text{discontinuous at } x=7}$ $= \int_2^{7} \frac{dx}{(x-1)(x-7)} + \int_7^{10} \frac{dx}{(x-1)(x-7)}$

improper.

$$\int_2^7 \frac{dx}{(x-1)(x-7)} = \lim_{t \to 7^-} \int_2^t \frac{-1/6}{x-1} + \frac{1/6}{x-7}\, dx$$

$$\frac{1}{(x-1)(x+7)} = \frac{A}{x-1} + \frac{B}{x+7}$$

$$1 = A(x-7) + B(x-1)$$

$x=7$: $1 = A \cdot 0 + B \cdot 6 \Rightarrow B = 1/6$

$x=1$: $1 = A(-6) + B \cdot 0 \Rightarrow A = -1/6$

$$= \lim_{t \to 7^-} \left[-\frac{1}{6}\ln|x-1| + \frac{1}{6}\ln|x-7| \right]_2^t$$

$$= \lim_{t \to 7^-} \left[\frac{1}{6}\ln\left|\frac{x-7}{x-1}\right| \right]_2^t$$

$$= \lim_{t \to 7^-} \left[\frac{1}{6}\ln\left|\frac{t-7}{t-1}\right| - \frac{1}{6}\ln(5) \right]$$

$\left|\frac{t-7}{t-1}\right| \to 0^+$ as $t \to 7^-$

$= \infty$

$\therefore$ integral diverges

LIATE

(b) (8 points) $\int_0^{\pi} e^{-t} \sin t \, dt$

IBP

$u = \sin t \qquad dv = e^{-t} dt$

$du = \cos t \, dt \qquad v = -e^{-t}$

$$= \left[-e^{-t} \sin t \right]_0^{\pi} - \int_0^{\pi} (-e^{-t}) \cos t \, dt$$

$$= [0 - 0] + \int_0^{\pi} e^{-t} \cos t \, dt$$

IBP

$u = \cos t \qquad dv = e^{-t} dt$

$du = -\sin t \qquad v = -e^{-t}$

$$= \left[-e^{-t} \cos t \right]_0^{\pi} - \int_0^{\pi} (-e^{-t})(-\sin t \, dt)$$

$$\Rightarrow \int_0^{\pi} e^{-t} \sin t \, dt = e^{-\pi} + 1 - \int_0^{\pi} e^{-t} \sin t \, dt$$

$$\Rightarrow 2\int_0^{\pi} e^{-t} \sin t \, dt = e^{-\pi} + 1$$

$$\Rightarrow \int_0^{\pi} e^{-t} \sin t \, dt = \frac{e^{-\pi} + 1}{2}$$

(c) (8 points) $\int_1^\infty \frac{\sqrt{x^2-1}}{x^4}dx$

Find $\int \frac{\sqrt{x^2-1}}{x^4}\,dx = \int \frac{\tan\theta}{\sec^4\theta}\cdot \sec\theta\cdot\tan\theta\cdot d\theta$

$= \int \frac{\tan^2\theta}{\sec^3\theta}\,d\theta$

Pattern: $\sqrt{x^2-1}$ trig sub.

$x = 1\cdot\sec\theta$

$dx = \sec\theta\cdot\tan\theta\, d\theta$

$x^4 = \sec^4\theta$

$\sqrt{x^2-1} = \tan\theta$

$= \int \sin^2\theta\cdot\cos\theta\, d\theta$

$u = \sin\theta \quad du = \cos\theta\, d\theta$

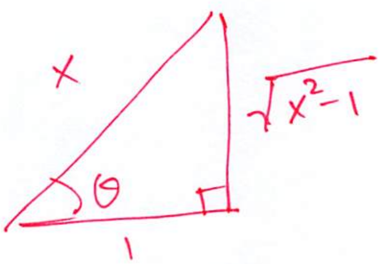
$= \int u^2\,du$

$= \frac{u^3}{3} + c$

$= \frac{(\sin\theta)^3}{3} + c$

$= \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + c$

$= \frac{1}{3}\left(\sqrt{1-\frac{1}{x^2}}\right)^3 + c$



$\sec\theta = \frac{hyp}{adj} = \frac{x}{1}$

$\sin\theta = \frac{opp}{hyp} = \frac{\sqrt{x^2-1}}{x}$

$\therefore \int_1^\infty \frac{\sqrt{x^2-1}}{x^4}\,dx = \lim_{t\to\infty}\int_1^t \frac{\sqrt{x^2-1}}{x^4}\,dx = \lim_{t\to\infty}\left[\frac{1}{3}\left(\sqrt{1-\frac{1}{x^2}}\right)^3\right]_1^t$

$= \lim_{t\to\infty}\left[\frac{1}{3}\left(\sqrt{1-\frac{1}{t^2}}\right)^3 - 0\right] = \frac{1}{3}\sqrt{1-0}^{\,3}$

$= \frac{1}{3}$

(d) (8 points) $\int_0^{\pi/4} \cot^3 x \, dx$

discontinuous at $x = 0$
improper

$= \lim_{t \to 0^+} \int_t^{\pi/4} \cot^3 x \, dx$

$\cot^3 x = \cot^2 x \cdot \cot x$
$= (\csc^2 x - 1) \cot x$
$= \csc^2 x \cdot \cot x - \cot x$

$= \lim_{t \to 0^+} \int_t^{\pi/4} \csc^2 x \cot x - \cot x \, dx$

$= \lim_{t \to 0^+} \left[\int_{\csc t}^{\sqrt{2}} u(-du) - \left[\ln|\sin x| \right]_t^{\pi/4} \right]$

$= \lim_{t \to 0^+} \left[\left[\frac{u^2}{2} \right]_{\sqrt{2}}^{\csc t} - \ln\left(\frac{1}{\sqrt{2}}\right) + \ln|\sin t| \right]$

$= \lim_{t \to 0^+} \left[\frac{\csc^2 t}{2} - 1 + \ln\sqrt{2} + \ln|\sin t| \right]$

∞ ∞

$= \infty$

diverges

6. Compute the sum of each of the convergent series below. Simplify your answers.

(a) (4 points) $\sum_{n=0}^{\infty} \frac{2^{2n+3}}{5^n}$ $= \sum_{n=0}^{\infty} \frac{4^n \cdot 8}{5^n} = \sum_{n=0}^{\infty} 8\left(\frac{4}{5}\right)^n = \frac{\text{first term}}{1-\text{ratio}}$

geometric $r = \frac{4}{5}$

$$= \frac{8 \cdot (4/5)^0}{1 - 4/5} \cdot \frac{5}{5}$$

$$= \frac{40}{5-4}$$

$$= 40$$

(b) (4 points) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ Hint. Identify it with a well-known series.

$= e^2$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

(c) (4 points) $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ telescoping

$$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2} = \frac{1/2}{n} - \frac{1/2}{n+2}$$

$1 = A(n+2) + Bn$

$n=-2$: $1 = A \cdot 0 + B(-2) \Longrightarrow B = -1/2$

$n=0$: $1 = A \cdot 2 + B \cdot 0 \Longrightarrow A = 1/2$

$$S_n = \left(\frac{1/2}{1} - \cancel{\frac{1/2}{3}}\right) + \left(\frac{1/2}{2} - \cancel{\frac{1/2}{4}}\right) + \left(\cancel{\frac{1/2}{3}} - \cancel{\frac{1/2}{5}}\right)$$
$$+ \left(\cancel{\frac{1/2}{4}} - \cancel{\frac{1/2}{6}}\right) + \left(\cancel{\frac{1/2}{5}} - \frac{1/2}{7}\right) + \left(\cancel{\frac{1/2}{6}} - \frac{1/2}{8}\right)$$
$$+ \cdots + \left(\cancel{\frac{1/2}{n-1}} - \frac{1/2}{n+1}\right) + \left(\cancel{\frac{1/2}{n}} - \frac{1/2}{n+2}\right)$$

$$S_n = \frac{1}{2} + \frac{1}{4} - \frac{1/2}{n+1} - \frac{1/2}{n+2} \longrightarrow \frac{1}{2} + \frac{1}{4} = \boxed{\frac{3}{4}}$$

7. For each of the following series, say whether they converge or diverge. For full credit, you must justify your solutions, and state clearly which test(s) you are using (if any).

(a) (7 points) $\sum_{n=3}^{\infty} \frac{1}{n \ln n}$

Integral test! $f(x) = \frac{1}{x \cdot \ln x}$

(1) $f(x)$ is positive for $x \geq 3$

(2) $f(x) = \frac{1}{x \cdot \ln x}$ is decreasing for $x \geq 3$

(3) $f(x)$ is cts on $[3, \infty)$

$$\int_3^{\infty} \frac{1}{x \cdot \ln x}\, dx = \lim_{t \to \infty} \int_3^{t} \frac{1}{x \cdot \ln x}\, dx$$

$u = \ln x$

$du = \frac{1}{x}\, dx$

$u(3) = \ln 3$

$u(t) = \ln t$

$$= \lim_{t \to \infty} \int_{\ln 3}^{\ln t} \frac{1}{u}\, du$$

$$= \lim_{t \to \infty} \ln|u| \Big]_{\ln 3}^{\ln t}$$

$$= \lim_{t \to \infty} \left(\ln \ln t - \ln \ln 3 \right)$$

$$= \infty$$

$\therefore$ series diverges

(b) (7 points) $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n}$

Divergence test:

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e \neq 0$$

series diverges

(c) (7 points) $\sum_{n=1}^{\infty} \frac{2n^2 + n^4}{n + n^6}$ Behaves like $\sum_{n=1}^{\infty} \frac{n^4}{n^6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$

try limit comparison test.

$$\lim_{n\to\infty} \frac{2n^2+n^4}{n+n^6} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \frac{2n^4+n^6}{n+n^6}$$

$$= \lim_{n\to\infty} \frac{2/n^2 + 1}{1/n^5 + 1}$$

$$= \frac{0+1}{0+1}$$

$$= 1$$

$\therefore$ Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

(p-series $p = 2 > 1$)

then $\sum_{n=1}^{\infty} \frac{2n^2+n^4}{n+n^6}$ also converges

(d) (7 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ factorials, try Ratio test.

$$\lim_{n\to\infty} \left| \frac{[(n+1)!]^2}{(2(n+1))!} \cdot \frac{(2n)!}{[n!]^2} \right|$$

$$= \lim_{n\to\infty} \left[\frac{(n+1)\cdot \cancel{n!}}{\cancel{n!}} \right]^2 \cdot \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} \cdot \frac{1/n^2}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{(1+1/n)^2}{(2+2/n)(2+1/n)}$$

$$= \frac{(1+0)^2}{(2+0)(2+0)}$$

$$= \frac{1}{4} < 1$$

$\therefore$ series converges

8. (a) (5 points) Find the radius of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{(x-5)^n}{\ln n}$$

Ratio test!

$$\left| \frac{(x-5)^{\cancel{n+1}}}{\ln(n+1)} \cdot \frac{\ln n}{\cancel{(x-5)^n}} \right| = |x-5| \cdot \frac{\ln n}{\ln(n+1)} \to |x-5|$$

$$|x-5| < 1 \iff -1 < x-5 < 1 \iff 4 < x < 6$$

Radius $= 1$.

(b) (5 points) For which values of x does the series

$$\sum_{n=2}^{\infty} \frac{(x-5)^n}{\ln n}$$

converge **conditionally**? Justify your answer.

Check endpts. $x=4$ & $x=6$ for conditional convergence.

$x=4$: $\displaystyle\sum_{n=2}^{\infty} \frac{(4-5)^n}{\ln n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges by AST.

$b_n = \frac{1}{\ln n}$ is pos, dec, and $b_n = \frac{1}{\ln n} \to 0$.

But $\displaystyle\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges (see below)

So the series converges conditionally at $x=4$.

$x=6$: $\displaystyle\sum_{n=2}^{\infty} \frac{(6-5)^n}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges by DCT since

$\frac{1}{\ln n} > \frac{1}{n}$ and $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ diverges (p-series $p = 1 \leq 1$)

Series cannot converge conditionally at $x=6$ since it diverges

9. (7 points) Express the function

$$f(x) = \frac{1+x^2}{1-x^2}$$

as a power series in x.

$$= \frac{1}{1-x^2} + x^2 \cdot \frac{1}{1-x^2}$$

$$= \left(1 + (x^2)^1 + (x^2)^2 + \cdots\right) + x^2\left(1 + (x^2)^1 + (x^2)^2 + \cdots\right)$$

$$= \left(1 + x^2 + x^4 + \cdots\right) + \left(x^2 + x^4 + x^6 + \cdots\right)$$

$$= 1 + 2x^2 + 2x^4 + 2x^6 + \cdots$$

10. (7 points) Find the Taylor series for the function

$$f(x) = x^3$$

centered at $a = 2$.

$f(x) = x^3 \Rightarrow f(2) = 8$

$f'(x) = 3x^2 \Rightarrow f'(2) = 12$

$f''(x) = 6x \Rightarrow f''(2) = 12$

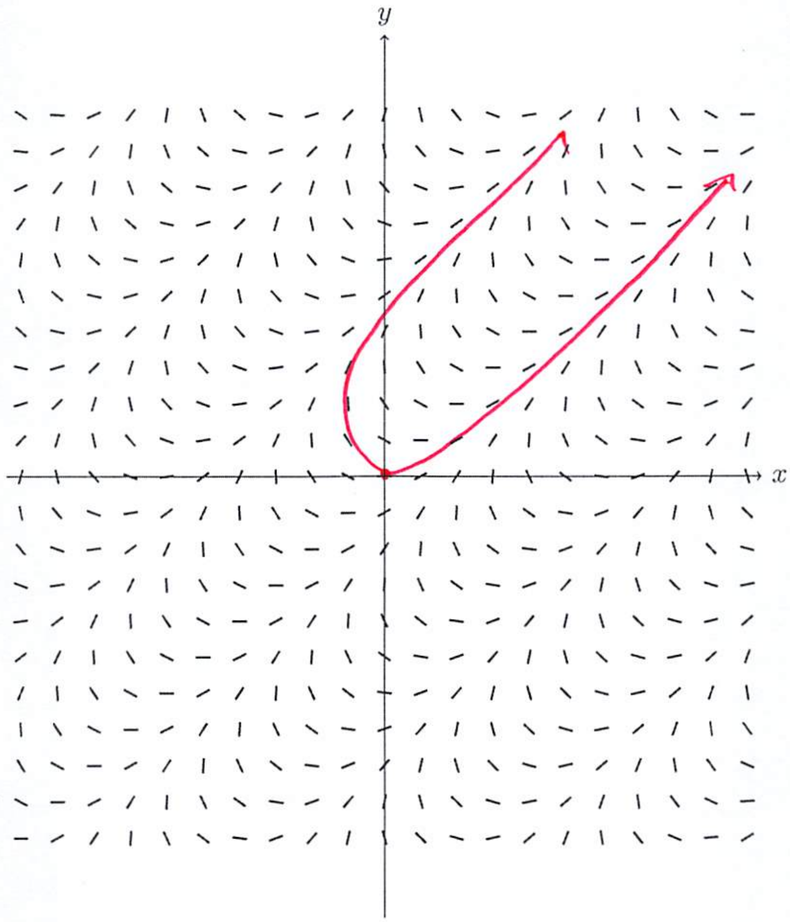
$f'''(x) = 6 \Rightarrow f'''(2) = 6$

$f^{(n)}(x) = 0 \Rightarrow f^{(n)}(2) = 0$

$n \geq 4$

$$T(x) = 8 + 12(x-2) + 12(x-2)^2 + 6(x-2)^3$$

11. Consider the slope field pictured below.



(a) (3 points) Which of the differential equations below matches this slope field?

(a) $\dfrac{dy}{dx} = e^{x-y}$, (b) $\dfrac{dy}{dx} = \tan(x-y)$ (c) $\dfrac{dy}{dx} = \ln(x-y)$, (d) $\dfrac{dy}{dx} = (x+y)^4$.

all slopes must be positive

Slopes do not exist for $x-y<0 \iff x<y$

all slopes must be positive.

(b) (3 points) Sketch all solutions to this differential equation that satisfy $y(0)=0$ on the slope field.

12. Solve the following differential equations. Either give the general solution, or solve for a particular solution satisfying the given initial conditions. Your solution must give an explicit formula for y for full credit.

(a) (8 points)

$$yy' - 2\cos^2 x = 0, \quad y(0) = -2.$$

Separable.

$$yy' = 2\cos^2 x$$

$$\Rightarrow \int y\,dy = \int 2\cos^2 x\,dx$$

$$\int y\,dy = \int \cancel{2}\cdot \frac{1+\cos 2x}{\cancel{2}}\,dx$$

$$\frac{y^2}{2} = x + \frac{1}{2}\sin 2x + C$$

$$y^2 = 2x + \sin 2x + C$$

$$y = \cancel{\pm}\sqrt{2x + \sin 2x + C}$$

$y(0) = -2$ take the minus.

$$-2 = -\sqrt{2\cdot 0 + \sin(0) + C} = -\sqrt{2+C}$$

$$4 = 2 + C$$

$$C = 2$$

$$\therefore y = -\sqrt{2x + \sin 2x + 2}$$

(b) (8 points) $xy' + y = 2x$. Linear

$$y' + \frac{1}{x}y = 2$$

$$p(x) = \frac{1}{x}$$

$$P(x) = \int \frac{1}{x}dx = \ln|x|$$

$$I(x) = e^{P(x)} = e^{\ln|x|} = |x|$$

$$y = \frac{1}{I(x)} \cdot \int 2 \cdot I(x)\, dx = \frac{1}{|x|}\int 2 \cdot |x|\, dx$$

$$= \frac{2}{x} \cdot \int x\, dx$$

$$= \frac{2}{x}\left(\frac{x^2}{2} + c\right)$$

$$= x + \frac{2c}{x}$$

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \quad \text{(true for } -1 < x < 1\text{)}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \quad \text{(true for all } x\text{)}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \quad \text{(true for } x < -1 \text{ and } x > 1\text{)}$$

- Pythagorean identities (true for all x where the functions involved are defined).

$$\sin^2(x) + \cos^2(x) = 1, \quad \tan^2(x) + 1 = \sec^2(x), \quad 1 + \cot^2(x) = \csc^2(x).$$

- Reduction of power formulas / double angle formulas for sine and cosine (true for all x).

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x)), \quad \sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

- Addition formulas for sine and cosine (true for all x and y).

$$\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$

$$\cos(x)\cos(y) = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$

$$\sin(x)\cos(y) = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

- Integrals of tangent, cotangent, and secant.

$$\int \tan(x)dx = -\ln|\cos(x)| + C$$

$$\int \cot(x)dx = \ln|\sin(x)| + C$$

$$\int \sec(x)dx = \ln|\sec(x) + \tan(x)| + C.$$

- Standard power series expansions (centered at $a = 0$).

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{(valid for all } x\text{)}.$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \text{(valid for all } x\text{)}.$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad \text{(valid for all } x\text{)}.$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad \text{(valid for } |x| < 1\text{)}.$$

$$(1+x)^m = \sum_{n=0}^{\infty} \frac{m(m-1)\cdots(m-n+1)}{n!} x^n \qquad \text{(valid for } |x| < 1\text{)}.$$

- Error estimate for approximations by Taylor polynomials.
  Say $f(x)$ is a function with derivatives of all orders on an interval $[b, c]$, and a is a point in $[b, c]$. Say $P_N(x)$ is the N^{th} Taylor polynomial for $f(x)$ centered at a, and $R_N(x) = f(x) - P_N(x)$ is the error when approximating $f(x)$ by $P_N(x)$. Then for all x in $[b, c]$

$$|R_N(x)| \leq \frac{M_{N+1}|x-a|^{N+1}}{(N+1)!},$$

  where M_{N+1} is the largest value taken by the $(N+1)^{\text{st}}$ derivative of $f(x)$ on $[b, c]$.