Math 242 Final

Name: ______________________________ **Section:** _______

Instructor: ______________________________

Question	Points	Score
1	21	
2	11	
3	8	
4	12	
5	13	
6	10	
7	9	
8	5	
9	5	
10	6	
Total:	100	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. For each of the following definite and indefinite integrals, evaluate it or show that it diverges.

(a) (6 points) $\int_0^{\pi/2} x \cos(x) dx$

(b) (7 points) $\int \frac{x^2}{(4-x^2)^{3/2}} dx$

(c) (8 points) $\int_3^\infty \frac{1}{x(2x-1)}dx$

2. For each of the following series decide if it converges or diverges and explain why.

(a) (5 points) $\sum_{n=1}^{\infty} \frac{n^4 + n^2}{n^5 + n}$

(b) (6 points) $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

3. For each of the following series, determine its sum.

(a) (4 points) $\sum_{n=0}^{\infty}(-1)^n \frac{1}{4^n}$

(b) (4 points) $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$

4. Find the derivative of each of the following functions.

(a) (6 points) $f(x) = 2^x \ln(x)$

(b) (6 points) $g(x) = \left(\sin^{-1}(5x)\right)^3$

5. Consider the following differential equation

$$y' - \frac{1}{5x}y = x, \quad x > 0.$$

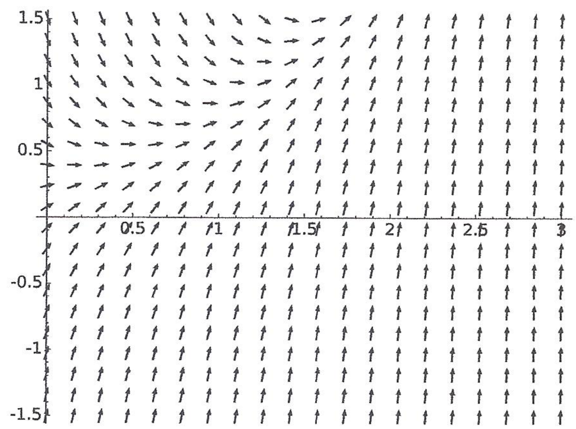
(a) (8 points) Find the general solution to this equation.

(b) (2 points) Find the particular solution given the initial condition $y(1) = 1$.

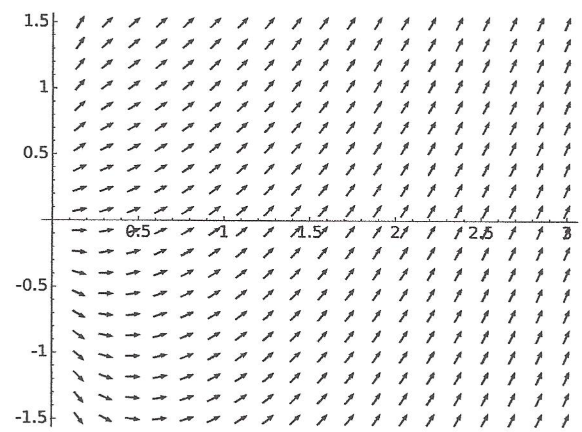
(c) (3 points) Which of the following plots represents the slope field of this differential equation? That is, of the equation

$$y' - \frac{1}{5x}y = x, \quad x > 0.$$

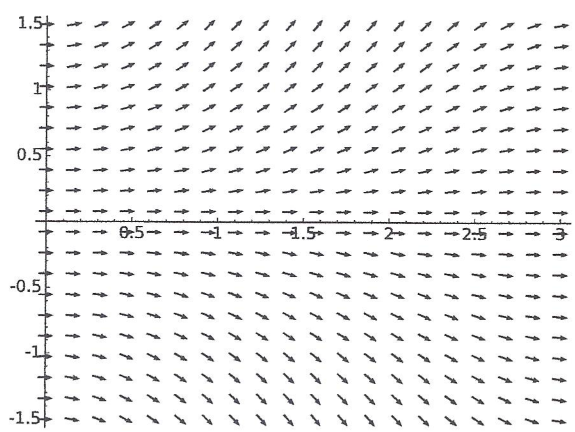
Circle your answer.



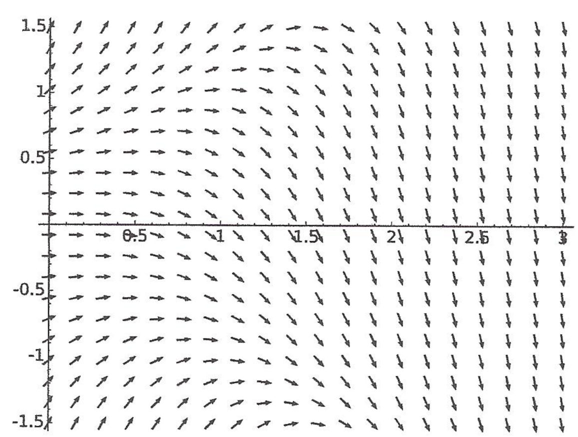
(I)



(II)



(III)



(IV)

6. In this problem, you will use numerical integration to estimate $\ln(2) = \int_1^2 \frac{dx}{x}$.

(a) (4 points) Graph the function $y = 1/x$ between $x = 1$ and $x = 2$. Draw on your graph the trapezoids used to apply the Trapezoidal Rule with $n = 2$. (So, your graph should have 2 trapezoids.)

(b) (4 points) Use the Trapezoidal Rule with $n = 2$ to estimate $\ln 2$.

(c) (2 points) Does the Trapezoidal Rule overestimate or underestimate $\ln 2$?

7. Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$.

(a) (7 points) Find its interval of convergence.

(b) (2 points) For what x does the series converge absolutely?

8. (5 points) Evaluate the following limit.

$$\lim_{t\to 0}(1+t)^{\frac{1}{2t}}$$

9. (5 points) Consider the order 2 Taylor polynomial for $\ln(1+x)$ centered at $a = 0$:

$$\ln(1+x) \approx x - \frac{x^2}{2}.$$

Use the Taylor remainder estimation theorem to estimate the error in this approximation when $|x| < 0.1$.

10. (6 points) What is the Taylor polynomial of order 3 for the function $f(x) = \sin(x)\cos(x)$ centered at $a = 0$?

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

- Trigonometric identities.

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1\\
1 + \tan^2 x &= \sec^2 x\\
1 + \cot^2 x &= \csc^2 x\\
\sin^2 x &= \frac{1}{2}(1 - \cos(2x))\\
\cos^2 x &= \frac{1}{2}(1 + \cos(2x))\\
\sin x \cos x &= \frac{1}{2}\sin(2x)\\
\sin x \sin y &= \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)\\
\cos x \cos y &= \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)\\
\sin x \cos y &= \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)
\end{aligned}$$

- Integrals of trigonometric functions.

$$\begin{aligned}
\int \tan x \; dx &= \ln|\sec x| + C\\
\int \cot x \; dx &= \ln|\sin x| + C\\
\int \sec x \; dx &= \ln|\sec x + \tan x| + C\\
\int \csc x \; dx &= -\ln|\csc x + \cot x| + C
\end{aligned}$$

- Trapezoidal Rule and Simpson's Rule.

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n \right)$$
$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n \right)$$

- Error estimates for Trapezoidal Rule and Simpson's Rule.

$$|E_T| \le \frac{M(b-a)^3}{12n^2}, \quad \text{where } |f''(x)| \le M \text{ for all } x \text{ in } [a,b]$$
$$|E_S| \le \frac{M(b-a)^5}{180n^4}, \quad \text{where } |f^{(4)}(x)| \le M \text{ for all } x \text{ in } [a,b]$$

- Famous Maclaurin series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (R = \infty)$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (R = \infty)$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (R = \infty)$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad (R = 1)$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad (R = 1)$$

- Error estimate for approximations by Taylor polynomials.

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(t)| \le M$ for all t between a and x.