Math 242 Final Spring 2017

Name: ______________________________

☐ Section 1, Thursday 10:30-11:20, Sita Benedict

☐ Section 2, Thursday 1:30-2:20, Sita Benedict

☐ Section 3, Thursday 10:30-11:20, David Yuen

☐ Section 4, Thursday 12:00-12:50, David Yuen

☐ Section 5, Friday 11:30-12:20, Achilles Beros

☐ Section 6, Friday 2:30-3:20, Achilles Beros

☐ Section 7, Friday 8:30-9:20, Piper Harron

☐ Section 8, Friday 9:30-10:20, Piper Harron

☐ Section 9, Friday 10:30-11:20, Les Wilson

☐ Section 10, Friday 1:30-2:20, Les Wilson

Solutions by D. Yuen

Page	Points	Score
2	8	
3	6	
4	6	
5	8	
6	8	
7	6	
8	9	
9	4	
10	8	
11	12	
12	10	
13	7	
14	10	
15	8	
Total:	110	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. Circle either true or false. You do not need to justify your answer.

   (a) (2 points) $\lim_{x \to +\infty} e^{3x} = +\infty$.

   $\boxed{\text{TRUE}}$ FALSE

   (b) (2 points) $\lim_{x \to -\infty} e^{3x} = 0$.

   $\boxed{\text{TRUE}}$ FALSE

   (c) (4 points) If f is a differentiable and one-to-one function, then

   $$(f^{-1})'(x) = \frac{-1}{f'(x)},$$

   provided the denominator is nonzero.

   Theorem is $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

   TRUE $\boxed{\text{FALSE}}$

2. For each of the following definite and indefinite integrals, evaluate it or show that it diverges.

(a) (6 points) $\int_0^1 2xe^x\,dx$

Integration by parts

$$\int 2x e^x dx \qquad u = 2x \quad dv = e^x dx$$
$$du = 2dx \quad v = e^x$$

$$= 2xe^x - \int 2e^x dx$$

$$= 2xe^x - 2e^x + C$$

$$= \left[2xe^x - 2e^x\right]_0^1$$

$$= \left[2e^1 - 2e^1\right] - \left[0 - 2e^0\right]$$

$$= 2$$

(b) (6 points) $\int \frac{x^2}{1+x^2}dx$

Solution #1 Long division

$$\begin{array}{r} 1 \\ x^2+1 \overline{)\, x^2 \quad} \\ \underline{x^2+1} \\ -1 \end{array}$$

$$\int \left(1 + \frac{-1}{1+x^2}\right)dx$$

$$= x - \arctan x + C$$

OR $\frac{x^2}{1+x^2} = \frac{x^2+1-1}{x^2+1}$

$$= \frac{x^2+1}{x^2+1} + \frac{-1}{x^2+1}$$

Solution #2 Trig sub

$x = \tan\theta$

$1+x^2 = 1+\tan^2\theta = \sec^2\theta$

$dx = \sec^2\theta \, d\theta$

$$\int \frac{\tan^2\theta}{\sec^2\theta} \sec^2\theta \, d\theta$$

$\arctan x = \theta$

$$= \int \tan^2\theta \, d\theta$$

$$= \int (\sec^2\theta - 1)\, d\theta$$

$$= \tan\theta - \theta + C$$

$$= x - \arctan x + C$$

(c) (8 points) $\int_{-1}^{1} \frac{3x-2}{x^2+x-12}dx$

$$\frac{3x-2}{(x+4)(x-3)} = \frac{A}{x+4} + \frac{B}{x-3}$$

$$3x-2 = A(x-3) + B(x+4)$$

Clever values

$x=3 \Rightarrow 7 = 0 + B7 \Rightarrow 1 = B$

$x=-4 \Rightarrow -14 = A(-7) + 0 \Rightarrow 2 = A$

$$\int_{-1}^{1}\left(\frac{2}{x+4} + \frac{1}{x-3}\right)dx$$

$$= 2\ln|x+4| + \ln|x-3| \Big|_{-1}^{1}$$

$$= 2\ln 5 + \ln 2 - (2\ln 3 + \ln 4)$$

$$= 2\ln 5 + \ln 2 - 2\ln 3 - \ln 4$$

(d) (8 points) $\int_1^{\infty} \frac{\ln(x)}{x^2}dx$

Integration by parts

$$\int \frac{\ln x}{x^2}dx \quad u = \ln x,\ dv = x^{-2}dx$$
$$du = \frac{1}{x}dx,\ v = -x^{-1}$$

$$= -x^{-1}\ln x - \int -x^{-1}\frac{1}{x}dx$$
$$= -\frac{\ln x}{x} + \int \frac{1}{x^2}dx$$
$$= -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$= \lim_{b\to\infty} \int_1^b \frac{\ln x}{x^2}dx$$

$$= \lim_{b\to\infty} \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[-\frac{\ln b}{b} - \frac{1}{b} + \frac{\ln 1}{1} + \frac{1}{1}\right]$$

$-\frac{1}{b} \to 0$

Use L'H Rule

$$= \lim_{b\to\infty} \frac{-\frac{1}{b}}{1} - 0 + 0 + 1$$

$$= 0 \quad + 1$$

$$= 1$$

3. (6 points) Find the derivative of $g(x) = (\sin^{-1}(x))^x$.

Logarithmic differentiation

$$\ln g(x) = x \ln(\sin^{-1}(x))$$

$\frac{d}{dx}$ both sides

$$\frac{1}{g(x)} g'(x) = 1 \ln(\sin^{-1}(x)) + x \frac{1}{\sin^{-1}(x)} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$g'(x) = g(x)\left(\ln(\sin^{-1}(x)) + \frac{x}{\sin^{-1}(x)\sqrt{1-x^2}} \right)$$

$$= (\sin^{-1}(x))^x \left(\ln(\sin^{-1}(x)) + \frac{x}{\sin^{-1}(x)\sqrt{1-x^2}} \right)$$

Alternate solution:

Rewrite $g(x) = e^{x \ln(\sin^{-1}(x))}$ and proceed,

4. For each, determine if the given limit exists and find it if it does (you must justify any use of l'Hôpital's rule).

(a) (4 points) $\lim_{x\to 0^+} \sqrt{x}\ln(x^3)$

Limit type $0 \cdot -\infty$. Use $AB = \frac{B}{1/A}$

$$= \lim_{x\to 0^+} \frac{\ln(x^3)}{x^{-1/2}} \quad \text{Type } \frac{-\infty}{\infty}$$

L'H Rule

$$\stackrel{\text{L'H Rule}}{=} \lim_{x\to 0^+} \frac{\frac{3x^2}{x^3}}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} \frac{3x^2}{x^3}(-2x^{3/2})$$

$$= \lim_{x\to 0^+} -6x^{1/2} = 0$$

(b) (5 points) $\lim_{x\to +\infty} x^{3/x}$

Set $L =$

Limit type ∞^0

Use logarithmic technique

$$\ln L = \lim_{x\to\infty} \ln(x^{3/x})$$

$$= \lim_{x\to\infty} \frac{3}{x}\ln x$$

$$= \lim_{x\to\infty} \frac{3\ln x}{x} \quad \text{Limit type } \frac{\infty}{\infty}$$

$$\stackrel{\text{L'H Rule}}{=} \lim_{x\to\infty} \frac{\frac{3}{x}}{1} = 0$$

Then $L = e^0 = \boxed{1}$

5. (4 points) Find an upper bound for the error (using the relevant formula from the formula sheet) when one uses the Trapezoidal rule with $n = 4$ to estimate $\int_{-1}^{1} e^{x^2}\,dx$. **(Note: you do not need to find the approximation, only an upper bound for the error).**

$f(x) = e^{x^2}$, $a = -1$, $b = 1$.

$f'(x) = e^{x^2} 2x$

$f''(x) = e^{x^2}(2x)(2x) + e^{x^2} 2 = (4x^2+2)e^{x^2}$

For $-1 \leq x \leq 1$,

$|f''(x)| = |(4x^2+2)e^{x^2}| \leq (4\cdot 1+2)e^{1} = 6e$

So $M = 6e$ is valid.

Then the error is at most

$$|E_T| \leq \frac{M(b-a)^3}{12n^2} = \frac{6e(2)^3}{12\cdot 4^2} = \boxed{\frac{e}{4}}$$

6. Circle either true or false. You do not need to justify your answer.

(a) (4 points) The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges but not absolutely. In other words, it converges conditionally.

$\boxed{\text{TRUE}}$ FALSE

(b) (4 points) The sum of the series $\sum_{n=2}^{\infty} \frac{2}{5^n}$ is $\frac{1}{10}$.

$\boxed{\text{TRUE}}$ FALSE

$$\frac{1^{ST}\text{term}}{1-\text{ratio}} = \frac{\frac{2}{25}}{1-\frac{1}{5}} = \frac{\frac{2}{25}}{\frac{4}{5}}$$

$$= \frac{2}{25}\cdot\frac{5}{4} = \frac{1}{10}$$

7. For each of the following series decide if it converges or diverges and explain why by explicitly stating which test(s) are used in your solution.

positive series (a) (6 points) $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$ $\Leftarrow$ Motivation: behaves like $\sum \frac{n}{n^2} = \sum \frac{1}{n}$

Solution #1 Regular comparison: $\frac{n+1}{n^2} \geq \frac{n}{n^2} = \frac{1}{n}$.

Since $\sum \frac{1}{n}$ diverges (p-series, $p=1$), then $\sum \frac{n+1}{n^2}$ diverges.

Solution #2 Limit compare with $\sum \frac{1}{n}$

$\lim_{n\to\infty} \frac{n+1}{n^2} / \frac{1}{n} = \lim_{n\to\infty} \frac{n^2+n}{n^2} \stackrel{LHR}{=} \lim_{n\to\infty} \frac{2n+1}{2n} \stackrel{L'HR}{=} \lim_{n\to\infty} \frac{2}{2}$

$= 1 \neq 0$. Since $\sum \frac{1}{n}$ diverges, then $\sum \frac{n+1}{n^2}$ diverges.

(b) (6 points) $\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n^2}$ positive series

Solution #1 $\frac{\tan^{-1}(n)}{n^2} \leq \frac{\pi/2}{n^2}$. Since $\sum \frac{\pi/2}{n^2}$

converges (p-series, $p=2>1$), then $\sum \frac{\tan^{-1}(n)}{n^2}$ converges,

Solution #2 Limit compare with $\sum \frac{1}{n^2}$.

$\lim_{n\to\infty} \frac{\tan^{-1}(n)}{n^2} / \frac{1}{n^2} = \lim_{n\to\infty} \tan^{-1}(n) = \frac{\pi}{2} \neq 0$.

Since $\sum \frac{1}{n^2}$ converges, then $\sum \frac{\tan^{-1}(n)}{n^2}$ converges.

8. Consider the power series $\sum_{n=1}^{\infty} \frac{(x-4)^n}{3^n\sqrt{n}}$.

(a) (2 points) What is the center of the power series?

4

(b) (6 points) What is radius of convergence of the power series?

Absolute Ratio Test: $\rho = \lim_{n\to\infty} \left| \frac{(x-4)^{n+1}}{3^{n+1}\sqrt{n+1}} \Big/ \frac{(x-4)^n}{3^n\sqrt{n}} \right| = \lim_{n\to\infty} \frac{|x-4|}{3} \frac{\sqrt{n}}{\sqrt{n+1}}$

$= \frac{1}{3}|x-4|$. Solve $\rho < 1$ to get $\frac{1}{3}|x-4| < 1$
$|x-4| < 3$

Radius is 3.

Solution #2 Absolute Root Test

$\rho = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-4)^n}{3^n\sqrt{n}}\right|} = \lim_{n\to\infty} \frac{|x-4|}{3\sqrt{\sqrt[n]{n}}} = \frac{|x-4|}{3\sqrt{1}}$

$= \frac{1}{3}|x-4|$. Same conclusion.

(c) (2 points) Does the power series converge absolutely at $x = 2$? Justify your answer.

~~the interval of convergence is~~
Since $x=2$ is within $|x-4|<3$, then yes the power series converges absolutely at $x=2$.

Alternate solution Just work on the series $\sum \frac{(2-4)^n}{3^n\sqrt{n}}$ directly for absolute convergence (see if $\sum \frac{2^n}{3^n\sqrt{n}}$ converges).

9. (7 points) Compute the Taylor polynomial of order 2 for the function $f(x) = \sqrt{x+4}$ centered at $x = 0$.

$f(x) = (x+4)^{1/2}$ $\quad f(0) = 4^{1/2} = 2$

$f'(x) = \frac{1}{2}(x+4)^{-1/2}$ $\quad f'(0) = \frac{1}{2} 4^{-1/2} = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$

$f''(x) = -\frac{1}{4}(x+4)^{-3/2}$ $\quad f''(0) = -\frac{1}{4} 4^{-3/2} = -\frac{1}{4}\cdot\frac{1}{8} = \frac{-1}{32}$

$$P_2(x) = 2 + \frac{1}{4}(x-0) + \frac{1}{2!}\left(-\frac{1}{32}\right)(x-0)^2$$

$$= 2 + \frac{1}{4}x - \frac{1}{64}x^2$$

10. (10 points) Find the general solution of the following differential equation

$$y' + \frac{1}{x}y = \frac{\sin^3(x)}{x}, \quad x > 0.$$

Linear, standard form

$P(x) = \frac{1}{x}$

Integrating factor is...

(+C not needed)

$$e^{\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$xy' + y = \sin^3(x)$$

$$(xy)' = \sin^3(x)$$

$$xy = \int \sin^3(x)dx$$

ODD power of sin

Let $w = \cos x$

$dw = -\sin x\,dx$

$1 - w^2 = \sin^2 x$

$$= \int \sin^2 x \, \sin x \, dx$$

$$= \int (1 - w^2)(-dw)$$

$$= -(w - \tfrac{1}{3}w^3) + C$$

$$= -\cos x + \tfrac{1}{3}\cos^3 x + C$$

$$xy = -\cos x + \frac{1}{3}\cos^3(x) + C$$

$$y = -\frac{\cos x}{x} + \frac{1}{3}\,\frac{\cos^3 x}{x} + C \cdot \frac{1}{x}$$

11. (8 points) Solve the initial value problem

$$y'' - 6y' + 8y = 0 \qquad y(0) = 0, \qquad y'(0) = 2$$

Characteristic equation is

$$r^2 - 6r + 8 = 0$$

$$(r-2)(r-4) = 0$$

$$r = 2, 4.$$

General solution is

$$y = C_1 e^{2x} + C_2 e^{4x}$$

$$y' = 2C_1 e^{2x} + 4C_2 e^{4x}$$

$$y(0)=0 \Rightarrow 0 = C_1 + C_2 \Rightarrow C_1 = -C_2$$

$$y'(0)=2 \Rightarrow 2 = 2C_1 + 4C_2$$

$$2 = 2(-C_2) + 4C_2$$

$$2 = 2C_2$$

$$1 = C_2 \quad \text{then} \quad C_1 = -1$$

$$y = -e^{2x} + e^{4x}$$

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

- Trigonometric identities.

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
1 + \tan^2 x &= \sec^2 x \\
1 + \cot^2 x &= \csc^2 x \\
\sin^2 x &= \frac{1}{2}(1 - \cos(2x)) \\
\cos^2 x &= \frac{1}{2}(1 + \cos(2x)) \\
\sin x \cos x &= \frac{1}{2}\sin(2x) \\
\sin x \sin y &= \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y) \\
\cos x \cos y &= \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y) \\
\sin x \cos y &= \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)
\end{aligned}$$

- Integrals of trigonometric functions.

$$\begin{aligned}
\int \tan x \, dx &= \ln|\sec x| + C \\
\int \cot x \, dx &= \ln|\sin x| + C \\
\int \sec x \, dx &= \ln|\sec x + \tan x| + C \\
\int \csc x \, dx &= -\ln|\csc x + \cot x| + C
\end{aligned}$$

- Trapezoidal Rule and Simpson's Rule.

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right)$$
$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

- Error estimates for Trapezoidal Rule and Simpson's Rule.

$$|E_T| \le \frac{M(b-a)^3}{12n^2}, \quad \text{where } |f''(x)| \le M \text{ for all } x \text{ in } [a,b]$$
$$|E_S| \le \frac{M(b-a)^5}{180n^4}, \quad \text{where } |f^{(4)}(x)| \le M \text{ for all } x \text{ in } [a,b]$$

- Famous Maclaurin series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (R = \infty)$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (R = \infty)$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (R = \infty)$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad (R = 1)$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad (R = 1)$$

- Error estimate for approximations by Taylor polynomials.

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(t)| \le M$ for all t between a and x.