Math 242 Final

Name: ______________________________ **Section:** _______

Instructor: ____________________________ **Recitation Time:** _________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	16	
7	17	
8	14	
9	14	
10	8	
11	7	
12	7	
Total:	133	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. (10 points) Evaluate $\int_0^{\pi/4} x \sin(5x)\, dx$

2. (10 points) Evaluate the following integral or show that it diverges.

$$\int_3^\infty \frac{1}{x(2x-1)}\,dx$$

3. (10 points) Evaluate the following integral. (Hint: Use trig substitution.)

$$\int \frac{1}{(1+x^2)^2}\,dx,$$

4. For each of the following series decide if it converges or diverges and explain why.

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n^4 - 2n^2}{n^5 + n}$

(b) (5 points) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^2}$

5. For each of the following series, determine its sum.

(a) (5 points) $\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^n}$

(b) (5 points) $\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$

6. Find the derivative of each of the following functions.

(a) (8 points) $f(x) = (\ln x)^x$

(b) (8 points) $g(x) = \left(\sin^{-1}(7x)\right)^4$

7. Consider the following differential equation

$$y' - \frac{1}{5x}y = x, \quad x > 0.$$

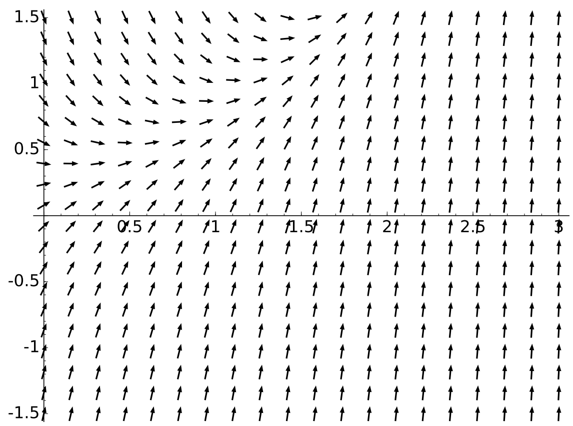
(a) (8 points) Find the general solution to this equation.

(b) (3 points) Find the particular solution given the initial condition $y(1) = 0$.

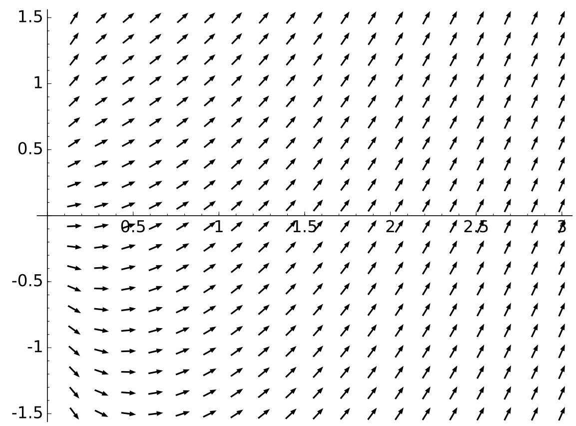
(c) (3 points) Which of the following plots represents the direction field of this differential equation? That is, of the equation

$$y' - \frac{1}{5x}y = x, \quad x > 0.$$

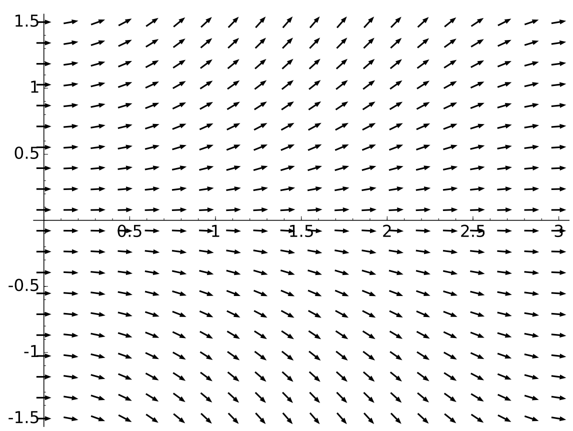
Circle your answer.



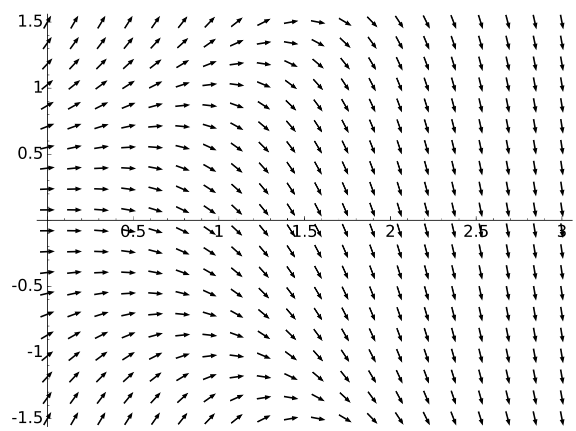
(I)



(II)



(III)

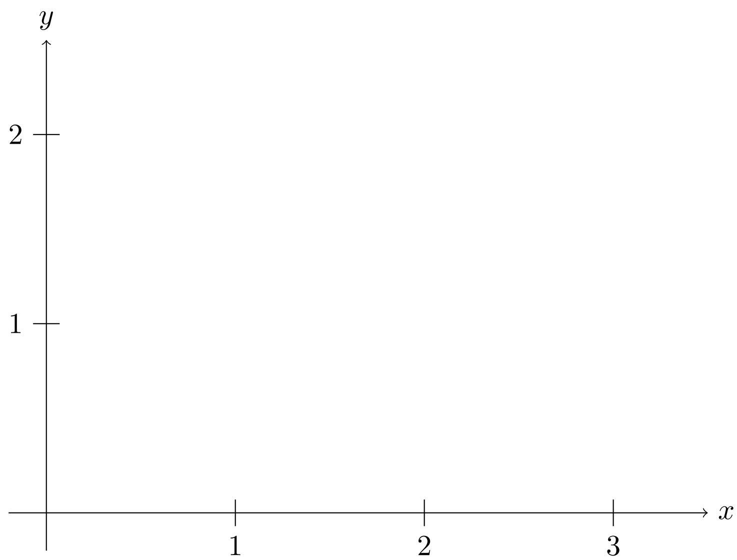


(IV)

(d) (3 points) Sketch the the particular solution to this differential equation that satisfies $y(1) = 0$ on the direction field you choose in part (c).

8. In this problem, you will use numerical integration to estimate $\ln(3) = \int_1^3 \frac{dx}{x}$. (The formula sheet has both the formula for numerical integration and the error estimates.)

(a) (4 points) Graph the function $y = 1/x$ between $x = 1$ and $x = 3$. Draw on your graph the trapezoids used to apply the Trapezoidal Rule with $n = 2$. (So, your graph should have 2 trapezoids.)



(b) (2 points) Does the Trapezoidal Rule overestimate or underestimate the value of $\ln(3)$?

(c) (4 points) Use the Trapezoidal Rule with $n = 2$ to estimate $\ln(3)$.

(d) (4 points) Use the error estimate to give an upper bound on the absolute value of the error.

9. Consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n 4^n}{\sqrt{n}} x^n$.

(a) (10 points) Find its interval of convergence.

(b) (4 points) For which values of x does the series converge absolutely?

10. (8 points) Evaluate the following limit.

$$\lim_{t \to 0} (1+t)^{\frac{1}{3t}}$$

11. (7 points) Consider the degree 2 Taylor polynomial for $\ln(1+x)$ centered at $a = 0$:

$$\ln(1+x) \approx x - \frac{x^2}{2}.$$

Use the Taylor remainder estimation theorem to estimate the error in this approximation when $|x| < 0.1$.

12. (7 points) What is the Taylor polynomial of degree 3 for the function $f(x) = \dfrac{1}{1+x^3}$ centered at $a = 0$?

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{x\sqrt{x^2-1}}$$

- Trigonometric identities.

$$\sin^2 x + \cos^2 x = 1 \qquad \sin(x+y) = \sin x\cos y + \cos x \sin y$$
$$1 + \tan^2 x = \sec^2 x \qquad \cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$1 + \cot^2 x = \csc^2 x \qquad \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\sin^2 x = \frac{1}{2}(1-\cos(2x))$$
$$\cos^2 x = \frac{1}{2}(1+\cos(2x))$$
$$\sin x \cos x = \frac{1}{2}\sin(2x)$$
$$\sin x \sin y = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$$
$$\cos x \cos y = \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)$$
$$\sin x \cos y = \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)$$

- Integrals of trigonometric functions.

$$\int \tan x \; dx = \ln|\sec x| + C$$
$$\int \cot x \; dx = \ln|\sin x| + C$$
$$\int \sec x \; dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x \; dx = -\ln|\csc x + \cot x| + C$$

- Trapezoidal Rule and Simpson's Rule.

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right)$$
$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

- Error estimates for Trapezoidal Rule and Simpson's Rule.

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}, \quad \text{where } |f''(x)| \leq M \text{ for all } x \text{ in } [a,b]$$
$$|E_S| \leq \frac{M(b-a)^5}{180n^4}, \quad \text{where } |f^{(4)}(x)| \leq M \text{ for all } x \text{ in } [a,b]$$

- Famous Maclaurin series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (R = \infty)$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (R = \infty)$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (R = \infty)$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad (R = 1)$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \qquad (R = 1)$$

- Error estimate for approximations by Taylor polynomials.

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(t)| \leq M$ for all t between a and x.