Math 242 Common Final Spring 2022

Name:

Solutions
by Kenny

Question	Points	Score
1	3	
2	10	
3	12	
4	10	
5	40	
6	32	
7	10	
8	6	
9	7	
10	6	
11	7	
12	7	
Total:	150	

- You may not use notes or calculators on the test.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- The last two pages are a formula sheet. You are welcome to remove this from the exam.
- Good luck!

1. (3 points) Consider the following values of $f(x)$ and $f'(x)$:

x	$f(x)$	$f'(x)$
0	1	-2
1	3	1
2	6	0
3	0	4

- Find $f^{-1}(1)$ $= x$

$\iff f(x) = 1$

so $x = 0$

- Find $(f^{-1})'(1)$ $= \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{-2}$

2. Compute the derivatives of the following functions. You do not have to simplify your answers.

(a) (4 points) $f(x) = \sin^{-1}(x+1)$.

$$f'(x) = \frac{1}{\sqrt{1-(x+1)^2}} \cdot 1$$

(b) (6 points) $f(x) = x^{3x}\ln(x)$.

$$\ln f(x) = \ln\left[x^{3x} \cdot \ln(x)\right] = 3x \cdot \ln x + \ln\ln x$$

$$\frac{f'(x)}{f(x)} = 3 \cdot \ln x + 3x \cdot \frac{1}{x} + \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$f'(x) = x^{3x} \cdot \ln x \left[3\ln x + 3 + \frac{1}{x\ln x}\right]$$

3. Compute the following limits. You must justify your solution using algebraic manipulations and / or l'Hôpital's rule for full credit.

(a) (4 points) $\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{e^n}$ Type $\frac{\infty}{\infty}$

$$\stackrel{H}{=} \lim_{n\to\infty} \frac{\frac{1}{2}n^{-1/2}}{e^n}$$

$$= \lim_{n\to\infty} \frac{1}{2n^{1/2}\cdot e^n} = 0.$$

(b) (4 points) $\displaystyle\lim_{N\to\infty} \sum_{k=0}^{N} \frac{3^{k+1}}{4^k} = \sum_{k=0}^{\infty} \frac{3^{k+1}}{4^k} = \sum_{k=0}^{\infty} 3\cdot\left(\frac{3}{4}\right)^k$

$$= \frac{3^{0+1}/4^0}{1-3/4} = \frac{3}{1-3/4}$$

$$= \frac{12}{4-3} = 12$$

(c) (4 points) $\displaystyle\lim_{x\to 0^+} (1+2x)^{1/x}$ Type 1^∞

L "

$$\Rightarrow \ln L = \lim_{x\to 0^+} \frac{\ln(1+2x)}{x} \quad \text{Type } \frac{0}{0}$$

$$\stackrel{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+2x}\cdot 2}{1} = \frac{2}{1+2\cdot 0} = 2$$

4. (a) (5 points) **Use the trapezoidal rule** with $n = 4$ to find an approximation to the integral

$$\int_0^4 e^{-2x} dx.$$

You do not need to simplify your answer. You will not receive credit for evaluating the integral.

$a=0, \quad b=4, \quad n=4, \quad \Delta x = \frac{b-a}{n} = 1$

x_i	0	1	2	3	4
$y_i = e^{-2x_i}$	1	e^{-2}	e^{-4}	e^{-6}	e^{-4}
T_4 coeffs.	1	2	2	2	1
	1	$2e^{-2}$	$2e^{-4}$	$2e^{-6}$	$1e^{-4}$

mult

$$T_4 = \frac{\Delta x}{2} \cdot \text{sum} = \frac{1}{2}\left(1 + 2e^{-2} + 2e^{-4} + 2e^{-6} + e^{-4}\right)$$

(b) (5 points) Using the error formula for the trapezoidal rule, estimate how far your answer to (a) is from the actual value of the integral. For full credit, you should give and justify your estimate for "M" in the formula.

$f(x) = e^{-2x}$

$f'(x) = -2e^{-2x}$

$f''(x) = 4e^{-2x}$

f'' is a positive decreasing function

$$|f''(x)| \le |f''(0)| = 4$$

we may use $M = 4$.

$$|E_T| \le \frac{4 \cdot (4-0)^3}{12 \cdot 4^2}$$

5. Compute the following integrals, or say if they diverge. For full credit, follow the instructions below.

- If the integral is improper, make sure any limiting arguments that you use are explicit.
- If make a substitution, make sure to express your final answer in the original variables.
- Simplify your answers.

(a) (10 points) $\int_1^e x^3 \ln x dx.$ $= \frac{x^4}{4} \cdot \ln x \Big|_1^e - \int_1^e \frac{x^3}{4} dx$

$u = \ln x \quad dv = x^3 dx$

$du = \frac{1}{x} dx \quad v = \frac{x^4}{4}$

$= \frac{e^4}{4} \cdot 1 - 0 - \left[\frac{x^4}{16} \right]_1^e$

$= \frac{e^4}{4} - \left(\frac{e^4}{16} - \frac{1}{16} \right)$

$= \frac{3e^4 + 1}{16}$

(b) (10 points) $\int_3^\infty \frac{1}{3x(x-1)} dx.$

$= \lim_{t \to \infty} \int_3^t \frac{1}{3x(x-1)} dx$

$\frac{1}{3x(x-1)} = \frac{A}{3x} + \frac{B}{x-1}$

$= \lim_{t \to \infty} \left[\int_3^t \frac{-1}{3x} + \frac{1/3}{x-1} dx \right]$

$1 = A(x-1) + B \cdot 3x$

$x = 1: \quad 1 = B \cdot 3 \Rightarrow B = 1/3$

$x = 0: \quad 1 = -A + 0 \Rightarrow A = -1$

$= \lim_{t \to \infty} \left[-\frac{1}{3} \ln|x| + \frac{1}{3} \ln|x-1| \right]_3^t$

$= \lim_{t \to \infty} \left[\frac{1}{3} \ln \left| \frac{x-1}{x} \right| \right]_3^t$

$= \lim_{t \to \infty} \frac{1}{3} \ln \left| \frac{t-1}{t} \right| - \frac{1}{3} \ln \frac{2}{3}$

$= -\frac{1}{3} \ln \frac{2}{3}.$

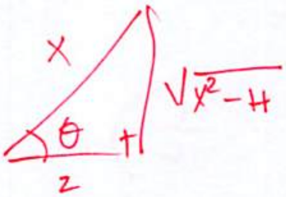
(c) (10 points) $\int \frac{1}{\sqrt{x^2-4}}dx.$ $= \int \frac{1}{\cancel{2\tan\theta}} \cdot \cancel{2}\sec\theta\,\cancel{\tan\theta}\,d\theta$

$x = 2\sec\theta$

$\sqrt{x^2-4} = 2\tan\theta$ $\qquad = \int \sec\theta \, d\theta$

$dx = 2\sec\theta \cdot \tan\theta \, d\theta$

$= \ln|\tan\theta + \sec\theta| + C$



$= \ln\left|\frac{\sqrt{x^2-4}}{2} + \frac{x}{2}\right| + C$

$\sec\theta = \frac{hyp}{adj} = \frac{x}{2}$

(d) (10 points) $\int \cos(x)\sin(2x)dx$.

$$= \int \frac{1}{2} \sin(2x - x) + \frac{1}{2} \sin(2x+x)\, dx$$

$$= \frac{1}{2} \int \sin x + \sin 3x \, dx$$

$$= \frac{1}{2} \left(-\cos x - \frac{1}{3} \cos 3x \right) + C$$

6. For each of the following series, say whether they converge or diverge. For full credit, you must justify your solutions, and state clearly which test(s) you are using (if any).

(a) (8 points) $\sum_{n=1}^{\infty} \frac{2}{1+e^{-n}}$.

$$\lim_{n \to \infty} \frac{2}{1+e^{-n}} = \frac{2}{1+0} = 2 \neq 0$$

Series diverges by divergence test

(b) (8 points) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{1+n}$. Alternating series test:

$$b_n = \frac{1}{1+n}$$

(i) $b_n > 0$

(ii) $b_{n+1} \leq b_n$ since $\frac{1}{1+n}$ is a decreasing function (numerator fixed & denominator grows)

(iii) $\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n+1} = 0$.

$\therefore$ Series converges

(c) (8 points) $\sum_{n=0}^{\infty} \frac{n^2}{(n+2)!}$. Ratio test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)^2}{(n+3)!} \cdot \frac{(n+2)!}{n^2} \right|$$

$$= \frac{(n+1)^2}{(n+3) \cdot n^2} = \frac{n^2 + 2n + 1}{n^3 + 3n^2} \longrightarrow 0 < 1$$

$\therefore$ Series converges

(d) (8 points) $\sum_{n=1}^{\infty} \frac{2}{\ln(n) + n^3}$. Comparison test

$$\frac{2}{\ln n + n^3} \leq \frac{2}{n^3}$$

Since $\sum_{n=1}^{\infty} \frac{2}{n^3}$ converges (p-series $p = 3$)

then $\sum_{n=1}^{\infty} \frac{2}{\ln n + n^3}$ converges.

7. Consider the power series in x

$$\sum_{n=1}^{\infty} \frac{4^n}{n}(x+1)^n.$$

(a) (7 points) Find the interval of convergence. Be sure to check the endpoints.

Ratio test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{4^{n+1}(x+1)^{n+1}}{n+1} \cdot \frac{n}{4^n(x+1)^n}\right|$

$= 4 \cdot |x+1| \cdot \frac{n}{n+1} \longrightarrow 4|x+1|$

$4|x+1| < 1 \iff |x+1| < 1/4 \iff -1/4 < x+1 < 1/4$

$\iff -5/4 < x+1 < -3/4$

$x = -5/4$: $\sum_{n=1}^{\infty} \frac{4^n}{n} \cdot (-5/4+1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by AST

but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-series $p=1$)

$x = -3/4$: $\sum_{n=1}^{\infty} \frac{4^n}{n} \cdot \left(-\frac{3}{4}+1\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (see above)

(b) (3 points) For which values of x does the series converge absolutely?

I.O.C : $[-5/4, -3/4)$

Absolute Convergence : $(-5/4, -3/4)$

11. (7 points) Solve the following differential equations for a particular solution satisfying the given initial conditions. Your solution must give an explicit formula for y for full credit.
$\frac{dy}{dx} = \tan(x)\, y, \quad y(0) = 2.$

$$\frac{1}{y}\, dy = \tan x \, dx$$

$$\int \frac{1}{y}\, dy = \int \tan x \, dx$$

$$\ln|y| = \ln|\sec x| + C$$

$$|y| = e^{C} \cdot e^{\ln|\sec x|} = e^{C} \cdot |\sec x|$$

$$y = \pm e^{C} \cdot |\sec x|$$

$$y = A \cdot |\sec x| \qquad (y=0 \text{ is a solution})$$

$$2 = y(0) = A \cdot 1 \implies A = 2.$$

$$y = 2 \cdot |\sec x|$$

12. (7 points) Find a general solution for the following differential equation. Your solution must give an explicit formula for y for full credit.
$y' + y = 2x$.

$P(x) = 1$

Integrating factor: $e^{\int 1\,dx} = e^x$

$$\Rightarrow \quad e^x y' + e^x y = 2xe^x$$

$$(e^x y)' = 2xe^x$$

$$e^x y = \int 2xe^x$$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$$e^x y = 2\left[xe^x - \int e^x dx\right]$$

$$= 2xe^x - 2e^x + C$$

$$\Rightarrow \quad y = e^{-x}(2xe^x - 2e^x + C)$$

8. (6 points) Find the **second** degree Taylor polynomial for $\tan^{-1}(x^3)$, centered at $a = 1$.

$$f(x) = \tan^{-1}(x^3) \Rightarrow f(1) = \tan^{-1}(1) = \pi/4$$

$$f'(x) = \frac{1}{1+x^6} \cdot 3x^2 \Rightarrow f'(1) = \frac{3}{2}$$

$$f''(x) = \frac{(1+x^6).6x - 3x^2(6x^5)}{(1+x^6)^2} \Rightarrow f''(1) = \frac{2.6-3.6}{4}$$

$$= -\frac{3}{2}$$

$$T_2(x) = \pi/4 + 3/2\,(x-1) + \frac{-3/2}{2!}(x-1)^2$$

9. (7 points) Find the **third** degree Taylor polynomial for $e^x \sin(x)$ centered at $a = 0$. Hint: you may find the expressions for famous Maclaurin series on the last page useful.

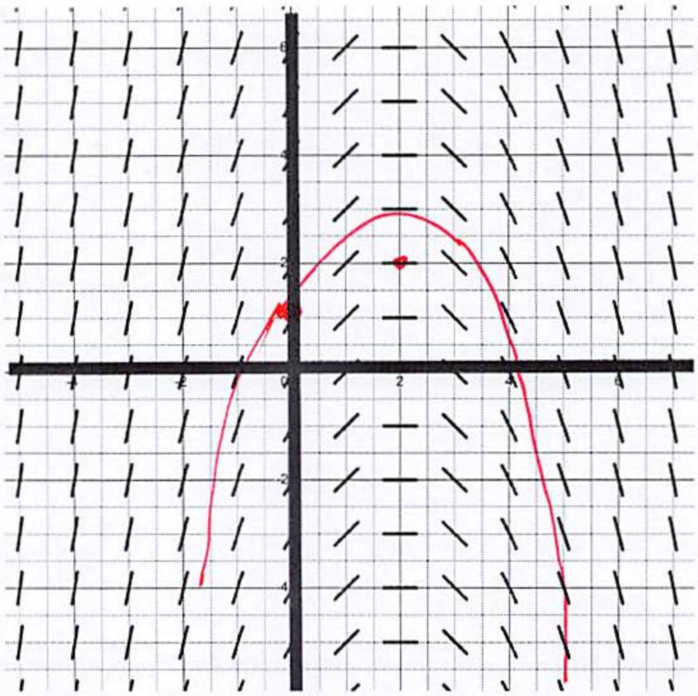
e^x

	0	x	0	$-x^3/6$	$\dots$
1	0	x	0	$-x^3/6$	
x	0	x^2	0	$-x^4/6$	
$\frac{x^2}{2}$	0	$\frac{x^3}{2}$	0		
$\frac{x^3}{6}$	0		0		
$\vdots$					

T_3

~~[illegible]~~

10. Consider the slope field pictured below.



(a) (3 points) Which of the differential equations below matches this slope field?

(a) $\frac{dy}{dx} = \frac{x}{y}$, (b) $\frac{dy}{dx} = 2 - x$, (c) $\frac{dy}{dx} = x^2 + y^2$, (d) $\frac{dy}{dx} = -y$.

(2,2) $=1$ $=0$ $=8$ $=-2$

(b) (3 points) Sketch the solution to this differential equation that satisfies $y(0) = 1$ on the slope field.

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

- Trigonometric identities.

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1\\
1+\tan^2 x &= \sec^2 x\\
1+\cot^2 x &= \csc^2 x\\
\sin^2 x &= \frac{1}{2}(1-\cos(2x))\\
\cos^2 x &= \frac{1}{2}(1+\cos(2x))\\
\sin x\cos x &= \frac{1}{2}\sin(2x)\\
\sin x\sin y &= \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)\\
\cos x\cos y &= \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y)\\
\sin x\cos y &= \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)
\end{aligned}$$

- Integrals of trigonometric functions.

$$\begin{aligned}
\int \tan x\ dx &= \ln|\sec x| + C\\
\int \cot x\ dx &= \ln|\sin x| + C\\
\int \sec x\ dx &= \ln|\sec x + \tan x| + C\\
\int \csc x\ dx &= -\ln|\csc x + \cot x| + C\\
\int \sin^2 x dx &= x/2 - (1/2)\sin x\cos x + C\\
\int \cos^2 x dx &= x/2 + (1/2)\sin x\cos x + C
\end{aligned}$$

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- Trapezoidal Rule and Simpson's Rule.

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right)$$
$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

- Error estimates for Trapezoidal Rule and Simpson's Rule.

$$|E_T| \leq \frac{M(b-a)^3}{12n^2}, \quad \text{where } |f''(x)| \leq M \text{ for all } x \text{ in } [a,b]$$
$$|E_S| \leq \frac{M(b-a)^5}{180n^4}, \quad \text{where } |f^{(4)}(x)| \leq M \text{ for all } x \text{ in } [a,b]$$

- Famous Maclaurin series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (R = \infty)$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (R = \infty)$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (R = \infty)$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad (R = 1)$$

- Error estimate for approximations by Taylor polynomials.

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(t)| \leq M$ for all t between a and x.