Math 242 Final, Spring 2023

Solutions by Kenny.

May 10, 2023, 12:00-2:00

Name: ____________________

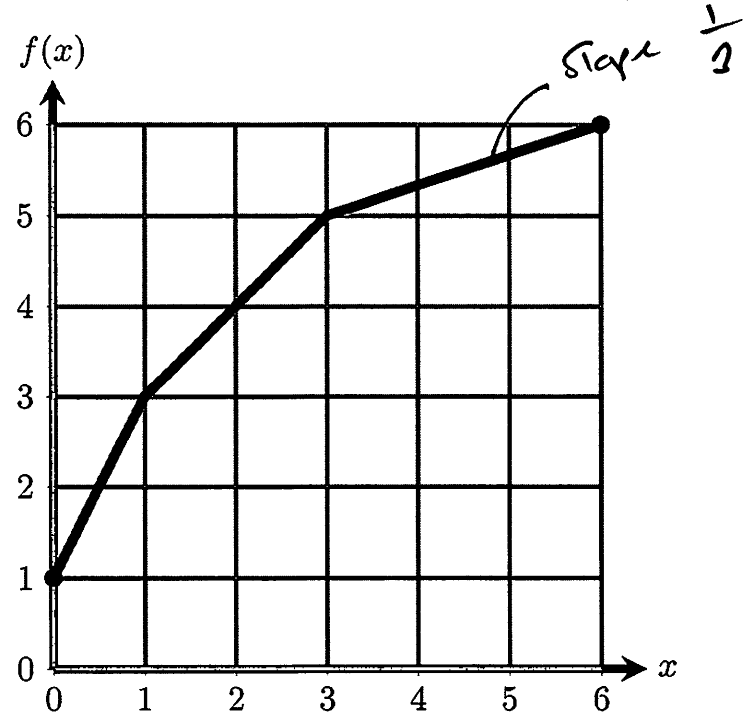
Draw a circle around your section number below.

	Instructor	TA	Recitation
1	Nha Truong	Drew Polakowski	Th 10:30-11:20
2	Nha Truong	Drew Polakowski	Th 1:30-2:20
3	Sébastien Bertrand	Saroj Niraula	Th 10:30-11:20
4	Sébastien Bertrand	Saroj Niraula	Th 12:00-12:50
5	Lyon Lanerolle	Dennis Le	F 11:30-12:20
6	Lyon Lanerolle	Dennis Le	F 2:30-3:20
7	Kenneth Corea	Samuel Miller	F 8:30-9:20
8	Kenneth Corea	Samuel Miller	F 9:30-10:20
9	Nicolas Antin	Irvin Chang	F 10:30-11:20
10	Nicolas Antin	Irvin Chang	F 1:30-2:20
11	Nicolas Antin	Aleksander Fedorynski	Th 8:00-8:50
12	Nicolas Antin	Aleksander Fedorynski	Th 9:00-9:50
13	Thomas Hangelbroek	Rico Vicente	W 1:30-2:20

Question	Points	Score
1	6	
2	12	
3	12	
4	30	
5	10	
6	21	
7	7	
8	16	
9	12	
10	8	
11	16	
Total:	150	

- You may not use notes.
- You may not use calculators.
- You may not use electronic devices or access the internet.
- Please ask if anything seems confusing or ambiguous.
- You must show all your work and make clear what your final solution is (e.g. by drawing a box around it).
- Organize your work neatly in the spaces provided and write neatly and legibly.
- Cross out any scratch work.
- You have 2 hours to complete this exam.
- Good luck!

1. The graph of a function $f(x)$ is below. The domain of f is $[0, 6]$. Answer the following questions about f.



(a) (2 points) What is the domain of f^{-1}?

$$\text{domain}(f^{-1}) = \text{range}(f) = [1, 6]$$

(b) (2 points) What is the value of $f^{-1}(5)$?

$$f^{-1}(5) = x \iff f(x) = 5$$
$$x = 3$$

(c) (2 points) What is the value of $(f^{-1})'(2)$?

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(4)} = \frac{1}{1/3} = 3$$

2. For each function below, calculate its derivative with respect to x.

(a) (6 points) $f(x) = \tan^{-1}(3^x)$

$$f'(x) = \frac{1}{1+(3^x)^2} \cdot 3^x \cdot \ln 3$$

(b) (6 points) $g(x) = e^{3x-4}\ln(x^2+x)$

$$g'(x) = e^{3x-4}(3)\cdot\ln(x^2+x) + e^{3x-4} \cdot \frac{1}{x^2+x} \cdot (2x+1)$$

3. Calculate the following limits.

(a) (6 points) $\lim_{x \to 0^+} x^{2/\ln(3x)}$

Type $0^{-\infty}$

Set $L = \lim_{x \to 0^+} x^{2/\ln(3x)}$

$\Rightarrow \ln L = \lim_{x \to 0^+} \frac{2}{\ln(3x)} \cdot \ln x$ Type $\frac{-\infty}{-\infty}$

$\overset{H}{=} \lim_{x \to 0^+} 2 \cdot \frac{1/x}{3/3x}$

$= \lim_{x \to \infty} 2 = 2$

$\therefore L = e^{\ln L} = e^2$

(b) (6 points) $\lim_{x \to 0} \frac{\ln(\cos x)}{x^2}$

Type $\frac{0}{0}$

$\overset{H}{=} \lim_{x \to 0} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{2x}$

$= \lim_{x \to 0} \frac{-\tan x}{2x}$ Type $\frac{0}{0}$

$\overset{H}{=} \lim_{x \to 0} \frac{-\sec^2 x}{2}$

$= -\frac{1}{2}$

(b) (10 points) $\int \frac{x^3}{\sqrt{9-x^2}}\,dx$ $= \int \frac{27\sin^3\theta}{\cancel{3\cos\theta}} \cdot \cancel{3\cos\theta}\,d\theta$

Pattern: $\sqrt{3^2 - x^2}$

use $x = 3\sin\theta$

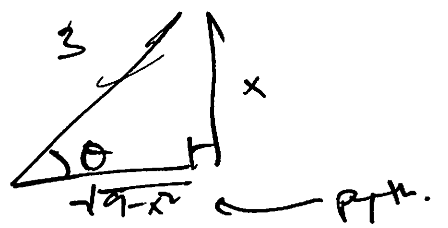
$x^3 = 27\sin^3\theta$

$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta}$

$= 3\sqrt{1-\sin^2\theta}$

$= 3\cos\theta$

$dx = 3\cos\theta\,d\theta$



$\sin\theta = \frac{opp}{hyp} = \frac{x}{3}$

$\Rightarrow \cos\theta = \frac{adj}{hyp} = \frac{\sqrt{9-x^2}}{3}$

$= 27\int \sin^3\theta\,d\theta$

ODD power sin

use $u = \cos\theta$

$du = -\sin\theta\,d\theta$

$\cancel{1-\sin^2\theta = 1-u^2}$

$\sin^2\theta = 1-\cos^2\theta$

$= 1-u^2$

$= 27\cdot\int \sin^2\theta \cdot \sin\theta\,d\theta$

$= 27\int (1-u^2)(-du)$

$= 27\left(\frac{u^3}{3} - u\right) + C$

$= 27\left(\frac{1}{3}(\cos\theta)^3 - \cos\theta\right) + C$

$= 27\left(\frac{1}{3}\left(\frac{\sqrt{9-x^2}}{3}\right)^3 - \left(\frac{\sqrt{9-x^2}}{3}\right)\right) + C$

4. Evaluate the following integrals. Note that some integrals below may be improper.

(a) (10 points) $\int_{-1}^{1} \frac{x^2}{\sqrt{x^3+1}}\,dx$

not defined @ $x=-1$

improper

$$= \lim_{t\to -1^+} \int_t^1 \frac{x^2}{\sqrt{x^3+1}}\,dx$$

$u = x^3+1$

$du = 3x^2\,dx$

$u(t) = t^3+1$

$u(1) = 1^3+1 = 2$

$$= \lim_{t\to -1^+} \int_{t^3+1}^{2} \frac{\frac{1}{3}\,du}{u^{1/2}}$$

$$= \lim_{t\to -1^+} \frac{1}{3}\int_{t^3+1}^{2} u^{-1/2}\,du$$

$$= \lim_{t\to -1^+} \frac{1}{3}\left[2u^{1/2}\right]_{t^3+1}^{2}$$

$$= \lim_{t\to -1^+} \frac{1}{3}\left[2\cdot\sqrt{2} - 2\sqrt{t^3+1}\right]$$

$$= \frac{1}{3}\left[2\sqrt{2} - 2\sqrt{-1+1}\right] = \frac{2}{3}\sqrt{2}$$

Log
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A
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E

(c) (10 points) $\int \frac{\ln(x-4)}{x^2}\,dx$ (Hint: Use integration by parts.)

$u = \ln(x-4)$ $\quad dv = \frac{1}{x^2}dx$

$du = \frac{1}{x-4}dx$ $\quad v = -\frac{1}{x}$

$= \ln(x-4)\left(-\frac{1}{x}\right) - \int \left(-\frac{1}{x}\right)\frac{1}{x-4}\,dx$

$= \frac{\ln(x-4)}{x} + \int \underbrace{\frac{1}{x(x-4)}}_{\text{PFD}}\,dx$

$\frac{1}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4} \implies 1 = A(x-4) + B(x)$

$x=4: \quad 1 = A\cdot 0 + B\cdot 4$

$\implies B = 1/4$

$x=0: \quad 1 = A(-4) + B\cdot 0$

$\implies A = -1/4$

$= \frac{\ln(x-4)}{x} + \int \frac{-1/4}{x} + \frac{1/4}{x-4}\,dx$

$= \frac{\ln(x-4)}{x} - \frac{1}{4}\ln|x| + \frac{1}{4}\ln|x-4| + C$

5. **Calculate the sum** of each series below. If it diverges, explain why.

(a) (5 points) $2 - \frac{4}{5} + \frac{8}{25} - \frac{16}{125} + \dots$ looks geometric

$$= 2\cdot\left(-\frac{2}{5}\right)^0 + 2\left(-\frac{2}{5}\right)^1 + 2\left(-\frac{2}{5}\right)^2 + \cdots$$

geometric $r = -2/5$

first term $= 2$

$$= \frac{2}{1-(-2/5)} = \frac{2}{1+2/5}\cdot\frac{5}{5} = \frac{10}{5+2} = \frac{10}{7}$$

(b) (5 points) $\sum_{n=2}^{\infty}\left(\frac{n-1}{n^2} - \frac{n}{(n+1)^2}\right)$ telescoping.

$$S_n = \left(\frac{1}{2^2} - \cancel{\frac{2}{3^2}}\right) + \left(\cancel{\frac{2}{3^2}} - \cancel{\frac{3}{4^2}}\right) + \left(\cancel{\frac{3}{4^2}} - \cancel{\frac{4}{5^2}}\right) + \cdots + \left(\cancel{\frac{n-1}{n^2}} - \frac{n}{(n+1)^2}\right)$$

$$S_n = \frac{1}{4} - \frac{n}{(n+1)^2} \longrightarrow \frac{1}{4} - 0 = \underbrace{\frac{1}{4}}_{\text{sum.}}$$

6. In each item below, determine whether the series converges or diverges. Justify your answers.

(a) (7 points) $\sum_{n=1}^{\infty} e^{-1/n}$

Divergence test

$$\lim_{n\to\infty} e^{-1/n} = e^{\lim_{n\to\infty}(-1/n)} = e^{0} = 1 \neq 0$$

Series diverges

(b) (7 points) $\sum_{n=0}^{\infty} \frac{n+n^3}{1+n^2+n^5}$

Direct comparison Test.

$$\frac{n+n^3}{1+n^2+n^5} \leq \frac{n^3+n^3}{1+n^5+n^2} \leq \frac{2n^3}{n^5} = 2\cdot\frac{1}{n^2}$$

Since the series $2\cdot\sum_{n=0}^{\infty}\frac{1}{n^2}$ converges (p-series $p=2>1$), then the series $\sum_{n=0}^{\infty}\frac{n+n^3}{1+n^2+n^5}$ converges.

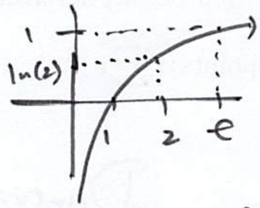
(c) (7 points) $\sum_{n=1}^{\infty} (\ln(2n+1) - \ln n)^n$ Try Root test.

$$\rho = \lim_{n \to \infty} \sqrt[n]{\left| \underbrace{(\ln(2n+1) - \ln(n))}_{\text{always positive}}^{n} \right|}$$

$$= \lim_{n \to \infty} \left(\ln(2n+1) - \ln(n) \right) \quad \text{Type } \infty - \infty$$

$$= \lim_{n \to \infty} \ln\left(\frac{2n+1}{n} \right) = \ln\left(\lim_{n \to \infty} \left(2 + \cancel{\tfrac{1}{n}}^{\,0} \right) \right) = \ln(2)$$

cts



since $\rho = \ln(2) < 1$ then the series converges (abs)

7. (7 points) Use a power series centered at 0 to integrate the following:

$$\int x e^{-x^3} \, dx$$

(Hint: Use a famous Maclaurin series.)

$$x e^{-x^3} = x \sum_{n=0}^{\infty} \frac{(-x^3)^n}{n!} = x \left(1 + (-x^3)^1 + \frac{(-x^3)^2}{2!} + \frac{(-x^3)^3}{3!} + \cdots \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{3n} \cdot x}{n!} = x - x \cdot x^3 + \frac{x x^6}{2!} - \frac{x \cdot x^9}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{n!} = x - x^4 + \frac{x^7}{2!} - \frac{x^{10}}{3!} + \cdots$$

$$\Rightarrow \int x e^{-x^3} \, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{(3n+2) \cdot n!} = \frac{x^2}{2} - \frac{x^5}{5} + \frac{x^8}{8 \cdot 2!} - \frac{x^{11}}{11 \cdot 3!} + \cdots + C$$

8. Consider the power series

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2}(x-5)^n.$$

(a) (7 points) Determine the radius of convergence of the series.

Ratio test:

$$\rho = \lim_{n\to\infty} \left| \frac{(n+1)+1}{(n+1)^2} \cdot (x-5)^{n+1} \cdot \frac{n^2}{n+1} \cdot \frac{1}{(x-5)^n} \right|$$

$$= \lim_{n\to\infty} \frac{n+2}{n+1} \cdot \left(\frac{n}{n+1}\right)^2 \cdot |x-5|$$

$\frac{n+2}{n+1} \to 1$, $\frac{n}{n+1} \to 1$

$$= |x-5|$$

$\rho < 1 \iff |x-5| < 1$

Radius

$-1 < x-5 < 1$

$4 < x < 6$ conv. abs.

(b) (7 points) Determine the interval of convergence of the series. Make sure to check the endpoints.

$x=4$: $\sum_{n=1}^{\infty} \frac{n+1}{n^2}(-1)^n$ AST: $b_n = \frac{n+1}{n^2}$ pos, dec, $b_n \to 0$

converges.

$x=6$: $\sum_{n=1}^{\infty} \frac{n+1}{n^2}(1)^n$ DCT: $\frac{n+1}{n^2} > \frac{n}{n^2} = \frac{1}{n}$

Since the harmonic series diverges, then the series diverges at $x=6$.

IOC: $[4,6)$

(c) (2 points) State where the series converges absolutely, where it converges conditionally, and where it diverges.

Since $\sum_{n=1}^{\infty} \left|(-1)^n \frac{n+1}{n^2}\right| = \sum_{n=1}^{\infty} \frac{n+1}{n^2}$ diverges, the series converges conditionally at $x=4$, converges absolutely for $4<x<6$, and diverges elsewhere

9. Let $f(x) = x^{1/3}$.

(a) (6 points) Calculate the degree-2 Taylor polynomial $T_2(x)$ of f centered at 1.

$f(x) = x^{1/3} \Rightarrow f(1) = 1^{1/3} = 1$

$f'(x) = \frac{1}{3}x^{-2/3} \Rightarrow f'(1) = \frac{1}{3}$.

$f''(x) = -\frac{2}{9}x^{-5/3} \Rightarrow f''(1) = -\frac{2}{9}$

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

$$= 1 + \frac{1}{3}(x-1) + \frac{-2/9}{2!}(x-1)^2$$

(b) (6 points) Estimate the remainder $\underbrace{|T_2(x) - f(x)|}_{R_2(x)}$ when $0.8 \leq x \leq 1.2$. (You do not need to simplify your answer.)

$\Leftrightarrow |x-1| < 0.2$

$f^{(3)}(x) = \underbrace{\frac{10}{27}x^{-8/3}}$

dec. on $0.8 \leq x \leq 1.2$

largest at $x = 0.8$

$|f^{(3)}(x)| \leq \underbrace{\frac{10}{27}(0.8)^{-8/3}}_{\text{this is } K}$

$\therefore |R_2(x)| \leq \frac{K}{3!}|x-1|^3$

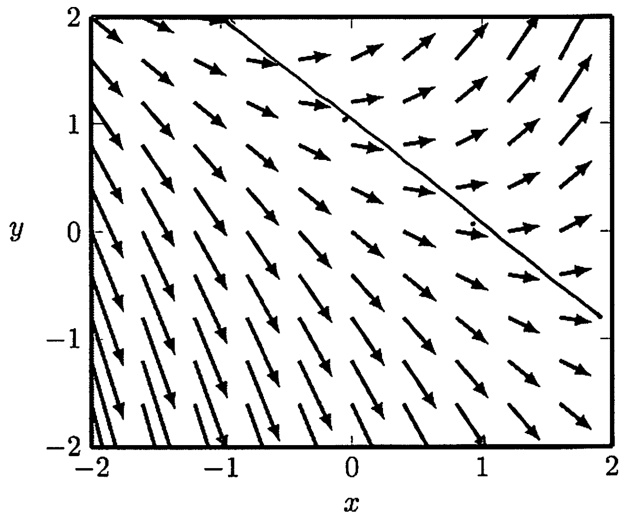
$\leq \frac{\frac{10}{27}(0.8)^{-8/3}(0.2)^3}{3!}$

10. (8 points) The direction fields for the following differential equations are shown below.

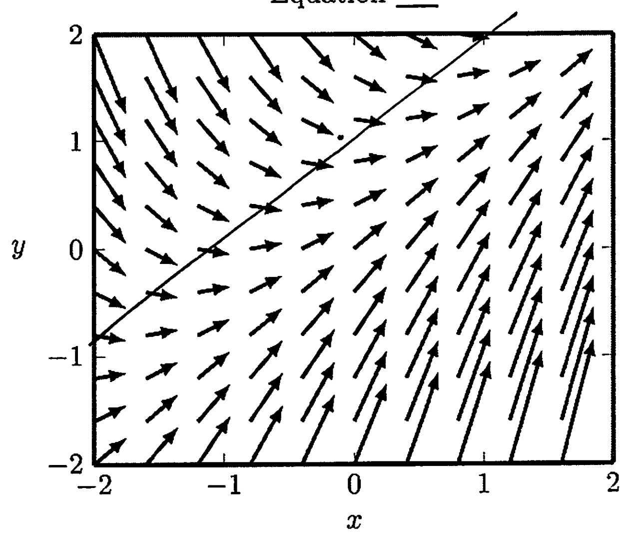
- Equation A: $y' = xy$ flat slopes on $xy = 0$
- Equation B: $y' = x + y - 1$ flat slope on $y = 1 - x$
- Equation C: $y' = x - y + 1$ flat slope on $y = 1 + x$
- Equation D: $y' = x^2 + y$ flat slope on $y = -x^2$

Match each equation to its direction field. Indicate your answer by writing the letter A, B, C, or D in the blank provided above each plot. You do not need to justify your answers.

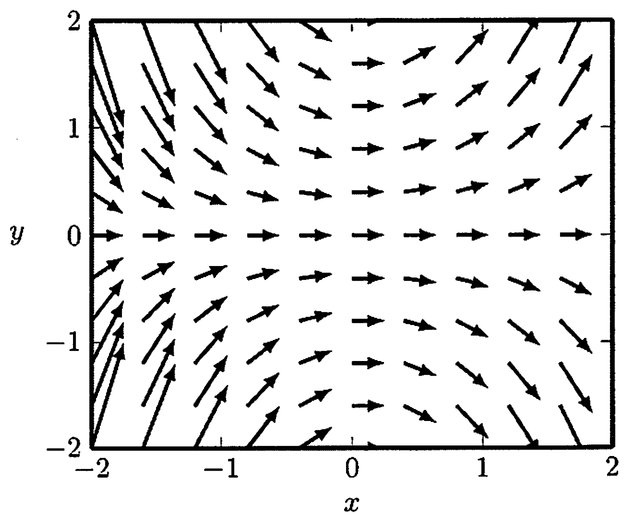
Equation B



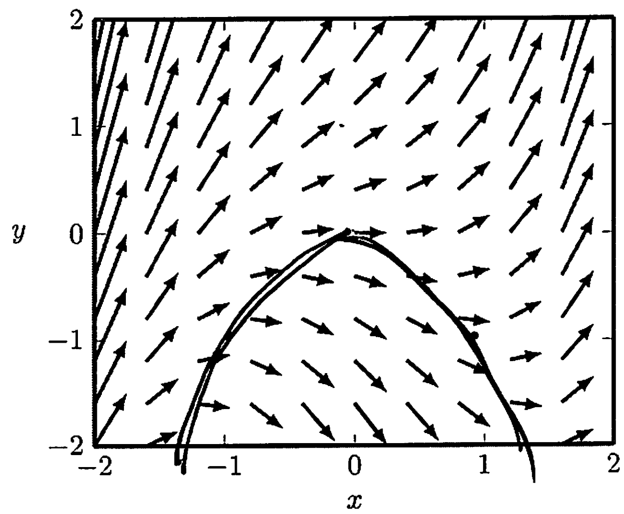
Equation C



Equation A



Equation D



11. Solve the following differential equations. If no initial condition is given, find the general solution. Otherwise, find the solution that satisfies the given initial condition. Your solution must give an explicit formula for y to receive full credit.

(a) (8 points) $y' - xy^2 = x, \quad y(0) = 1$

not linear, try separable.

$$\frac{dy}{dx} = x + xy^2 = x(1+y^2)$$

$$\Rightarrow \frac{dy}{1+y^2} = x\,dx$$

$$\Rightarrow \int \frac{dy}{1+y^2} = \int x\,dx$$

$$\arctan y = \frac{x^2}{2} + c$$

$$y = \tan\left(\frac{x^2}{2} + c\right)$$

$$y(0) = 1 \Rightarrow 1 = \tan(0 + c) = \tan(c)$$

$$\Rightarrow \tan\left(\frac{\pi}{4}\right) = 1 \quad c = \frac{\pi}{4}$$

$$\therefore y = \tan\left(\frac{x^2}{2} + \frac{\pi}{4}\right).$$

(b) (8 points) $y' + \underbrace{(\cos x)}_{p(x)}y = \underbrace{\sin x \cos x}_{q(x)}$ Linear

$$P(x) = \int p(x)\,dx = \int \cos x \, dx = \sin x$$

$$I(x) = e^{P(x)} = e^{\sin x}$$

$$\therefore\ y = e^{-\sin x} \cdot \int e^{\sin x} \cdot \sin x \cdot \cos x \, dx$$

$$u = \sin x$$
$$du = \cos x \, dx$$

$$= e^{-\sin x} \int e^{u} \cdot u \, du \qquad \text{I.B.P.}$$

$$\begin{array}{ccc} u & & e^u \\ & + & \\ 1 & & e^u \\ & - & \\ 0 & & e^u \end{array}$$

$$= e^{-\sin x}\left(ue^{u} - e^{u} + c\right)$$

$$= e^{-\sin x}\left(\sin x\, e^{\sin x} - e^{\sin x} + c\right)$$

Formula sheet

- Derivatives of inverse trigonometric functions.

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

- Trigonometric identities.

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
1 + \tan^2 x &= \sec^2 x \\
1 + \cot^2 x &= \csc^2 x \\
\sin^2 x &= \frac{1}{2}(1 - \cos(2x)) \\
\cos^2 x &= \frac{1}{2}(1 + \cos(2x)) \\
\sin x \cos x &= \frac{1}{2}\sin(2x) \\
\sin x \sin y &= \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y) \\
\cos x \cos y &= \frac{1}{2}\cos(x-y) + \frac{1}{2}\cos(x+y) \\
\sin x \cos y &= \frac{1}{2}\sin(x-y) + \frac{1}{2}\sin(x+y)
\end{aligned}$$

- Integrals of trigonometric functions.

$$\begin{aligned}
&\int \tan x \; dx = \ln|\sec x| + C \\
&\int \cot x \; dx = \ln|\sin x| + C \\
&\int \sec x \; dx = \ln|\sec x + \tan x| + C \\
&\int \csc x \; dx = -\ln|\csc x + \cot x| + C \\
&\int \sin^2 x dx = x/2 - (1/2)\sin x \cos x + C \\
&\int \cos^2 x dx = x/2 + (1/2)\sin x \cos x + C
\end{aligned}$$

- Trapezoidal Rule and Simpson's Rule.

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right)$$
$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

- Error estimates for Trapezoidal Rule and Simpson's Rule.

$$|E_T| \le \frac{M(b-a)^3}{12n^2}, \quad \text{where } |f''(x)| \le M \text{ for all } x \text{ in } [a,b]$$
$$|E_S| \le \frac{M(b-a)^5}{180n^4}, \quad \text{where } |f^{(4)}(x)| \le M \text{ for all } x \text{ in } [a,b]$$

- Famous Maclaurin series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (R = \infty)$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad (R = \infty)$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \qquad (R = \infty)$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \qquad (R = 1)$$

- Error estimate for approximations by Taylor polynomials.

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(t)| \le M$ for all t between a and x.