14. **N** A pond containing 1,000,000 gal of water is initially free of a certain undesirable chemical (see Problem 17 of Section 1.1). Water containing 0.01 g/gal of the chemical flows into the pond at a rate of 300 gal/h, and water also flows out of the pond at the same rate. Assume that the chemical is uniformly distributed throughout the pond.

a. Let $Q(t)$ be the amount of the chemical in the pond at time t. Write down an initial value problem for $Q(t)$.

b. Solve the problem in part a for $Q(t)$. How much chemical is in the pond after 1 year?

c. At the end of 1 year the source of the chemical in the pond is removed; thereafter pure water flows into the pond, and the mixture flows out at the same rate as before. Write down the initial value problem that describes this new situation.

d. Solve the initial value problem in part c. How much chemical remains in the pond after 1 additional year (2 years from the beginning of the problem)?

e. How long does it take for $Q(t)$ to be reduced to 10 g?

f. **G** Plot $Q(t)$ versus t for 3 years.

1.3 Classification of Differential Equations

The main purposes of this book are to discuss some of the properties of solutions of differential equations and to present some of the methods that have proved effective in finding solutions or, in some cases, in approximating them. To provide a framework for our presentation, we describe here several useful ways of classifying differential equations. Mastery of this vocabulary is essential to selecting appropriate solution methods and to describing properties of solutions of differential equations that you encounter later in this book — and in the real world.

Ordinary and Partial Differential Equations. One important classification is based on whether the unknown function depends on a single independent variable or on several independent variables. In the first case, only ordinary derivatives appear in the differential equation, and it is said to be an **ordinary differential equation.** In the second case, the derivatives are partial derivatives, and the equation is called a **partial differential equation.**

All the differential equations discussed in the preceding two sections are ordinary differential equations. Another example of an ordinary differential equation is

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t), \tag{1}$$

for the charge $Q(t)$ on a capacitor in a circuit with capacitance C, resistance R, and inductance L; this equation is derived in Section 3.7. Typical examples of partial differential equations are the heat conduction equation

$$\alpha^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \tag{2}$$

and the wave equation

$$a^2\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2}. \tag{3}$$

Here, α^2 and a^2 are certain physical constants. Note that in both equations (2) and (3) the dependent variable u depends on the two independent variables x and t. The heat conduction equation describes the conduction of heat in a solid body, and the wave equation arises in a variety of problems involving wave motion in solids or fluids.

Systems of Differential Equations. Another classification of differential equations depends on the number of unknown functions that are involved. If there is a single function to be determined, then one differential equation is sufficient. However, if there are two or more unknown functions, then a system of differential equations is required. For example, the Lotka-Volterra, or predator-prey, equations are important in ecological modeling. They have the form

$$\frac{dx}{dt} = ax - \alpha xy$$
$$\frac{dy}{dt} = -cy + \gamma xy, \tag{4}$$

where $x(t)$ and $y(t)$ are the respective populations of the prey and predator species. The positive constants a, α, c, and γ are based on empirical observations and depend on the particular pair of species being studied. Systems of equations are discussed in Chapters 7 and 9; in particular, the Lotka-Volterra equations are examined in Section 9.5. In some areas of application it is not unusual to encounter very large systems containing hundreds, or even many thousands, of differential equations.

Order. The **order** of a differential equation is the order of the highest derivative that appears in the equation. The equations in the preceding sections are all first-order equations, whereas equation (1) is a second-order equation. Equations (2) and (3) are also second-order partial differential equations. More generally, the equation

$$F(t, u(t), u'(t), \dots, u^{(n)}(t)) = 0 \tag{5}$$

is an ordinary differential equation of the n^{th} order. Equation (5) expresses a relation between the independent variable t and the values of the function u and its first n derivatives $u', u'', \dots, u^{(n)}$. It is convenient and customary in differential equations to write y for $u(t)$, with $y', y'', \dots, y^{(n)}$ standing for $u'(t), u''(t), \dots, u^{(n)}(t)$. Thus equation (5) is written as

$$F(t, y, y', \dots, y^{(n)}) = 0. \tag{6}$$

For example,

$$y''' + 2e^t y'' + yy' = t^4 \tag{7}$$

is a third-order differential equation for $y = u(t)$. Occasionally, other letters will be used instead of t and y for the independent and dependent variables; the meaning should be clear from the context.

We assume that it is always possible to solve a given ordinary differential equation for the highest derivative, obtaining

$$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)}). \tag{8}$$

This is mainly to avoid the ambiguity that may arise because a single equation of the form (6) may correspond to several equations of the form (8). For example, the equation

$$(y')^2 + ty' + 4y = 0 \tag{9}$$

leads to the two equations

$$y' = \frac{-t + \sqrt{t^2 - 16y}}{2} \quad \text{or} \quad y' = \frac{-t - \sqrt{t^2 - 16y}}{2}. \tag{10}$$

Linear and Nonlinear Equations. A crucial classification of differential equations is whether they are linear or nonlinear. The ordinary differential equation

$$F(t, y, y', \dots, y^{(n)}) = 0$$

is said to be **linear** if F is a linear function of the variables $y, y', \dots, y^{(n)}$; a similar definition applies to partial differential equations. Thus the general linear ordinary differential equation of order n is

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t). \tag{11}$$

Most of the equations you have seen thus far in this book are linear; examples are the equations in Sections 1.1 and 1.2 describing the falling object and the field mouse population. Similarly, in this section, equation (1) is a linear ordinary differential equation and equations (2) and (3) are linear partial differential equations. An equation that is not of the form (11) is a **nonlinear** equation. Equation (7) is nonlinear because of the term yy'. Similarly, each equation in the system (4) is nonlinear because of the terms that involve the product of the two unknown functions xy.

A simple physical problem that leads to a nonlinear differential equation is the oscillating pendulum. The angle $\theta = \theta(t)$ that an oscillating pendulum of length L makes with the vertical direction (see Figure 1.3.1) satisfies the equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0, \tag{12}$$

whose derivation is outlined in Problems 22 through 24. The presence of the term involving $\sin(\theta)$ makes equation (12) nonlinear.

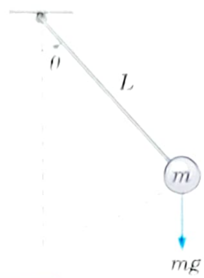

FIGURE 1.3.1 An oscillating pendulum.

The mathematical theory and methods for solving linear equations are highly developed. In contrast, for nonlinear equations the theory is more complicated, and methods of solution are less satisfactory. In view of this, it is fortunate that many significant problems lead to linear ordinary differential equations or can be approximated by linear equations. For example, for the pendulum, if the angle θ is small, then $\sin(\theta) \cong \theta$ and equation (12) can be approximated by the linear equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0. \tag{13}$$

This process of approximating a nonlinear equation by a linear one is called **linearization**; it is an extremely valuable way to deal with nonlinear equations. Nevertheless, there are many physical phenomena that simply cannot be represented adequately by linear equations. To study these phenomena, it is essential to deal with nonlinear equations.

In an elementary text it is natural to emphasize the simpler and more straightforward parts of the subject. Therefore, the greater part of this book is devoted to linear equations and various methods for solving them. However, Chapters 8 and 9, as well as parts of Chapter 2, are concerned with nonlinear equations. Whenever it is appropriate, we point out why nonlinear equations are, in general, more difficult and why many of the techniques that are useful in solving linear equations cannot be applied to nonlinear equations.

Solutions. A **solution** of the n^{th} order ordinary differential equation (8) on the interval $\alpha < t < \beta$ is a function ϕ such that $\phi', \phi'', \dots, \phi^{(n)}$ exist and satisfy

$$\phi^{(n)}(t) = f\left(t, \phi(t), \phi'(t), \dots, \phi^{(n-1)}(t)\right) \tag{14}$$

for every t in $\alpha < t < \beta$. Unless stated otherwise, we assume that the function f of equation (8) is a real-valued function, and we are interested in obtaining real-valued solutions $y = \phi(t)$.

Recall that in Section 1.2 we found solutions of certain equations by a process of direct integration. For instance, we found that the equation

$$\frac{dp}{dt} = \frac{p}{2} - 450 \tag{15}$$

has the solution

$$p(t) = 900 + ce^{t/2}, \tag{16}$$

where c is an arbitrary constant.

It is often not so easy to find solutions of differential equations. However, if you find a function that you think may be a solution of a given equation, it is usually relatively easy to determine whether the function is actually a solution: just substitute the function into the equation.

For example, in this way it is easy to show that the function $y_1(t) = \cos(t)$ is a solution of

$$y'' + y = 0 \tag{17}$$

for all t. To confirm this, observe that $y_1'(t) = -\sin(t)$ and $y_1''(t) = -\cos(t)$; then it follows that $y_1''(t) + y_1(t) = 0$. In the same way you can easily show that $y_2(t) = \sin(t)$ is also a solution of equation (17).

Of course, this does not constitute a satisfactory way to solve most differential equations, because there are far too many possible functions for you to have a good chance of finding the correct one by a random choice. Nevertheless, you should realize that you can verify whether any proposed solution is correct by substituting it into the differential equation. This can be a very useful check; it is one that you should make a habit of considering.

Some Important Questions. Although for the differential equations (15) and (17) we are able to verify that certain simple functions are solutions, in general we do not have such solutions readily available. Thus a fundamental question is the following: Does an equation of the form (8) always have a solution? The answer is "No." Merely writing down an equation of the form (8) does not necessarily mean that there is a function $y = \phi(t)$ that satisfies it. So, how can we tell whether some particular equation has a solution? This is the question of *existence* of a solution, and it is answered by theorems stating that under certain restrictions on the function f in equation (8), the equation always has solutions. This is not a purely theoretical concern for at least two reasons. If a problem has no solution, we would prefer to know that fact before investing time and effort in a vain attempt to solve the problem. Further, if a sensible physical problem is modeled mathematically as a differential equation, then the equation should have a solution. If it does not, then presumably there is something wrong with the formulation. In this sense an engineer or scientist has some check on the validity of the mathematical model.

If we assume that a given differential equation has at least one solution, then we may need to consider how many solutions it has, and what additional conditions must be specified to single out a particular solution. This is the question of *uniqueness*. In general, solutions of differential equations contain one or more arbitrary constants of integration, as does the solution (16) of equation (15). Equation (16) represents an infinity of functions corresponding to the infinity of possible choices of the constant c. As we saw in Section 1.2, if p is specified at some time t, this condition will determine a specific value for c; even so, we have not yet ruled out the possibility that there may be other solutions of equation (15) that also have the prescribed value of p at the prescribed time t. As in the question of existence of solutions, the issue of uniqueness has practical as well as theoretical implications. If we are fortunate enough to find a solution of a given problem, and if we know that the problem has a unique solution, then we can be sure that we have completely solved the problem. If there may be other solutions, then perhaps we should continue to search for them.

A third important question is: Given a differential equation of the form (8), can we actually determine a solution, and if so, how? Note that if we find a solution of the given equation, we have at the same time answered the question of the existence of a solution. However, without knowledge of existence theory we might, for example, use a computer to find a numerical approximation to a "solution" that does not exist. On the other hand, even though we may know that a solution exists, it may be that the solution is not expressible in terms of the usual elementary functions—polynomial, trigonometric, exponential, logarithmic, and hyperbolic functions. Unfortunately, this is the situation for most differential equations. Thus, we discuss both elementary methods that can be used to obtain exact solutions of certain relatively simple problems, and also methods of a more general nature that can be used to find approximations to solutions of more difficult problems.

Technology Use in Differential Equations. Technology provides many extremely valuable tools for the study of differential equations. For many years computers have been used to execute numerical algorithms, such as those described in Section 2.7 and in Chapter 8, to construct numerical approximations to solutions of differential equations. These algorithms have been refined to an extremely high level of generality and efficiency. A few lines of computer code, written in a high-level programming language and executed (often within a fraction of a second) on a relatively inexpensive computer, tablet, or smartphone, suffice to approximate to a high degree of accuracy the solutions of a wide range of differential equations. More sophisticated routines are also readily available. These routines combine the ability to handle very large and complicated systems with numerous diagnostic features that alert the user to possible problems as they are encountered.

The usual output from a numerical algorithm is a table of numbers, listing selected values of the independent variable and the corresponding values of the dependent variable. With appropriate software it is easy to display the solution of a differential equation graphically, whether the solution has been obtained numerically or as the result of an analytical procedure of some kind. Such a graphical display is often much more illuminating and helpful in understanding and interpreting the solution of a differential equation than a table of numbers or a complicated analytical formula. In the not so distant past, graphical representations of solutions of differential equations was possible only with the purchase of special-purpose software packages. Today, high quality web-based graphical tools are easily, and freely, accessible on a laptop computer, smartphone, or other portable device. Improvements in performance and access will continue to improve. The increased power and sophistication of modern smartphones, tablets, and other mobile devices have brought powerful computational and graphical capability within the reach of individual students. Several of our current favorite utilities are listed in the references at the end of this chapter. You should consider, in the light of your own circumstances, how best to take advantage of the available computing resources. You will surely find it enlightening to do so.

Another aspect of computer use that is very relevant to the study of differential equations is the availability of extremely powerful and general software packages that can perform a wide variety of mathematical operations. Among these are Maple, Mathematica, and MATLAB, each of which can be used on various kinds of computational platforms ranging from smartphones to massively parallel computers. All three of these packages can execute extensive numerical computations and have versatile graphical facilities. For example, they can perform the analytical steps involved in solving many differential equations, often in response to a single command. Anyone who expects to deal with differential equations in more than a superficial way should become familiar with at least one of these products and explore the ways in which it can be used.

For you, the student, these computing resources have an effect on how you should study differential equations. To become confident in using differential equations, it is essential to understand how the solution methods work, and this understanding is achieved, in part, by working out a sufficient number of examples in detail. However, eventually you should plan to utilize appropriate computational tools to complete many of the routine (often repetitive) details, while you focus on the proper formulation of the problem and on the interpretation of the solution. Our viewpoint is that you should always try to use the best methods and tools available for each task. In particular, you should strive to combine numerical, graphical, and analytical methods so as to attain maximum understanding of the behavior of the solution and of the underlying process that the problem models. You should also remember that some tasks can best be done with pencil and paper, while others require the use of some sort of computational technology. Good judgment, and experience, is often needed in selecting an effective combination.

Historical Background, Part III: Recent and Ongoing Advances. The numerous differential equations that resisted solution by analytical means led to the investigation of methods of numerical approximation (see Chapter 8). By 1900 fairly effective numerical integration methods had been devised, but their implementation was severely restricted by the need to execute the computations by hand or with very primitive computing equipment. Since World War II the development of increasingly powerful and versatile computers has vastly enlarged the range of problems that can be investigated effectively by numerical methods. Extremely refined and robust numerical integrators were developed during the same period and now are readily available, even on smartphones and other mobile devices. These technological advances have brought the ability to solve a great many significant problems within the reach of individual students.

Another characteristic of modern differential equations is the creation of geometric or topological methods, especially for nonlinear equations. The goal is to understand at least the qualitative behavior of solutions from a geometrical, as well as from an analytical, point of view. If more detailed information is needed, it can usually be obtained by using numerical approximations. An introduction to geometric methods appears in Chapter 9. We conclude this brief historical review with two examples that illustrate how computational and real-world experiences have motivated important analytical and theoretical discoveries.

In 1834 John Scott Russell (1808–1882), a Scottish civil engineer, was conducting experiments to determine the most efficient design for canal boats when he noticed that "when the boat suddenly stopped" the water being pushed by the boat "accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it [the boat] behind, [the water] rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water."[6] Many mathematicians did not believe that the solitary traveling waves reported by Russell existed. These objections were silenced when the doctoral dissertation of Dutch mathematician Gustav de Vries (1866–1934) included a nonlinear partial differential equation model for water waves in a shallow canal. Today these equations are known as the Korteweg-de Vries (KdV) equations. Diederik Johannes Korteweg (1848–1941) was de Vries's thesis advisor. Unknown to either Korteweg or de Vries, their Korteweg-de Vries model appeared as a footnote ten years earlier in French mathematician Joseph Valentin Boussinesq's (1842–1929) 680-page treatise *Essai sur la théorie des eaux courantes*. The work of Boussinesq and of Korteweg and deVries remained largely unnoticed until two Americans, physicist Norman J. Zabusky (1929–) and mathematician Martin David Kruskal (1925–2006), used computer simulations to discover, in 1965, that all solutions of the KdV equations eventually consist of a finite set of localized traveling waves. Today, nearly 200 years after Russell's observations and 50 years after the computational experiments of Zabusky and Kruskal, the study of "solitons" remains an active area of differential equations research. Other early contributors to nonlinear wave propagation include David Hilbert (German, 1862–1943), Richard Courant (German-American, 1888–1972), and John von Neumann (Hungarian-American, 1903–1957); we will encounter some of these ideas again in Chapter 9.

Computational results were also an essential element in the discovery of "chaos theory." In 1961, Edward Lorenz (1917–2008), an American mathematician and meteorologist at the Massachusetts Institute of Technology, was developing weather prediction models when he observed different results upon restarting a simulation in the middle of the time period using previously computed results. (Lorenz restarted the computation with three-digit approximate solutions, not the six-digit approximations that were stored in the computer.) In 1976 the Australian mathematician Sir Robert M. May (1938–) introduced and analyzed the logistic map, showing that there are special values of the problem's parameter where the solutions undergo drastic changes. The common trait that small changes in the problem produce large changes in the solution is one of the defining characteristics of chaos. May's logistic map is discussed in more detail in Section 2.9. Other classical examples of what we now recognize as "chaos" include the work by French mathematician Henri Poincaré (1854–1912) on planetary motion and the studies of turbulent fluid flow by Soviet mathematician Andrey Nikolaevich Kolmogorov (1903–1987), American mathematician Mitchell Feigenbaum (1944–), and many others. In addition to these and other classical examples of chaos, new examples continue to be found.

Solitons and chaos are just two of many examples where computers, and especially computer graphics, have given a new impetus to the study of systems of nonlinear differential equations. Other unexpected phenomena (Section 9.8), such as strange attractors (David Ruelle, Belgium, 1935–) and fractals (Benoit Mandelbrot, Poland, 1924–2010), have been

[6]"Report on Waves," in *Proceedings of the Fourteenth Meeting of the British Association for the Advancement of Science*, 1845, pp. 311–390, plus plates 47–57, http://www.macs.hw.ac.uk/~chris/Scott-Russell/SR44.pdf.

discovered, are being intensively studied, and are leading to important new insights in a variety of applications. Although it is an old subject about which much is known, the study of differential equations in the twenty-first century remains a fertile source of fascinating and important unsolved problems.

Problems

In each of Problems 1 through 4, determine the order of the given differential equation; also state whether the equation is linear or nonlinear.

1. $t^2\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + 2y = \sin(t)$
2. $(1+y^2)\dfrac{d^2y}{dt^2} + t\dfrac{dy}{dt} + y = e^t$
3. $\dfrac{d^4y}{dt^4} + \dfrac{d^3y}{dt^3} + \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + y = 1$
4. $\dfrac{d^2y}{dt^2} + \sin(t+y) = \sin(t)$

In each of Problems 5 through 10, verify that each given function is a solution of the differential equation.

5. $y'' - y = 0;\quad y_1(t) = e^t,\quad y_2(t) = \cosh(t)$
6. $y'' + 2y' - 3y = 0;\quad y_1(t) = e^{-3t},\quad y_2(t) = e^t$
7. $ty' - y = t^2;\quad y = 3t + t^2$
8. $y'''' + 4y''' + 3y = t;\quad y_1(t) = t/3,\quad y_2(t) = e^{-t} + t/3$
9. $t^2y'' + 5ty' + 4y = 0,\quad t > 0;\quad y_1(t) = t^{-2},\quad y_2(t) = t^{-2}\ln(t)$
10. $y' - 2ty = 1;\quad y = e^{t^2}\displaystyle\int_0^t e^{-s^2}\,ds + e^{t^2}$

In each of Problems 11 through 13, determine the values of r for which the given differential equation has solutions of the form $y = e^{rt}$.

11. $y' + 2y = 0$
12. $y'' + y' - 6y = 0$
13. $y''' - 3y'' + 2y' = 0$

In each of Problems 14 and 15, determine the values of r for which the given differential equation has solutions of the form $y = t^r$ for $t > 0$.

14. $t^2y'' + 4ty' + 2y = 0$
15. $t^2y'' - 4ty' + 4y = 0$

In each of Problems 16 through 18, determine the order of the given partial differential equation; also state whether the equation is linear or nonlinear. Partial derivatives are denoted by subscripts.

16. $u_{xx} + u_{yy} + u_{zz} = 0$
17. $u_{xxxx} + 2u_{xxyy} + u_{yyyy} = 0$
18. $u_t + uu_x = 1 + u_{xx}$

In each of Problems 19 through 21, verify that each given function is a solution of the given partial differential equation.

19. $u_{xx} + u_{yy} = 0;\quad u_1(x,y) = \cos(t)\cosh(y),$ $u_2(x,y) = \ln(x^2+y^2)$
20. $\alpha^2u_{xx} = u_t;\quad u_1(x,t) = e^{-\alpha^2 t}\sin(x),$ $u_2(x,t) = e^{-\alpha^2\lambda^2 t}\sin(\lambda x),\quad \lambda$ a real constant
21. $a^2u_{xx} = u_{tt};\quad u_1(x,t) = \sin(\lambda x)\sin(\lambda at),$ $u_2(x,t) = \sin(x-at),\quad \lambda$ a real constant
22. Follow the steps indicated here to derive the equation of motion of a pendulum, equation (12) in the text. Assume that the rod is rigid and weightless, that the mass is a point mass, and that there is no friction or drag anywhere in the system.

    a. Assume that the mass is in an arbitrary displaced position, indicated by the angle θ. Draw a free-body diagram showing the forces acting on the mass.

    b. Apply Newton's law of motion in the direction tangential to the circular arc on which the mass moves. Then the tensile force in the rod does not enter the equation. Observe that you need to find the component of the gravitational force in the tangential direction. Observe also that the linear acceleration, as opposed to the angular acceleration, is $Ld^2\theta/dt^2$, where L is the length of the rod.

    c. Simplify the result from part b to obtain equation (12) in the text.

23. Another way to derive the pendulum equation (12) is based on the principle of conservation of energy.

    a. Show that the kinetic energy T of the pendulum in motion is

    $$T = \frac{1}{2}mL^2\left(\frac{d\theta}{dt}\right)^2.$$

    b. Show that the potential energy V of the pendulum, relative to its rest position, is

    $$V = mgL(1-\cos(\theta)).$$

    c. By the principle of conservation of energy, the total energy $E = T + V$ is constant. Calculate dE/dt, set it equal to zero, and show that the resulting equation reduces to equation (12).

24. A third derivation of the pendulum equation depends on the principle of angular momentum: The rate of change of angular momentum about any point is equal to the net external moment about the same point.

    a. Show that the angular momentum M, or moment of momentum, about the point of support is given by $M = mL^2d\theta/dt$.

    b. Set dM/dt equal to the moment of the gravitational force, and show that the resulting equation reduces to equation (12). Note that positive moments are counterclockwise.