15. $ty' + (t+1)y = 2te^{-t}, \quad y(1) = a, \quad t > 0$

16. $(\sin(t))y' + (\cos(t))y = e^t, \quad y(1) = a, \quad 0 < t < \pi$

17. **G** Consider the initial value problem

$$y' + \frac{1}{2}y = 2\cos(t), \qquad y(0) = -1.$$

Find the coordinates of the first local maximum point of the solution for $t > 0$.

18. **N** Consider the initial value problem

$$y' + \frac{2}{3}y = 1 - \frac{1}{2}t, \qquad y(0) = y_0.$$

Find the value of y_0 for which the solution touches, but does not cross, the t-axis.

19. Consider the initial value problem

$$y' + \frac{1}{4}y = 3 + 2\cos(2t), \qquad y(0) = 0.$$

a. Find the solution of this initial value problem and describe its behavior for large t.

b. **N** Determine the value of t for which the solution first intersects the line $y = 12$.

20. Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin(t), \qquad y(0) = y_0$$

remains finite as $t \to \infty$.

21. Consider the initial value problem

$$y' - \frac{3}{2}y = 3t + 2e^t, \qquad y(0) = y_0.$$

Find the value of y_0 that separates solutions that grow positively as $t \to \infty$ from those that grow negatively. How does the solution that corresponds to this critical value of y_0 behave as $t \to \infty$?

22. Show that all solutions of $2y' + ty = 2$ [equation (41) of the text] approach a limit as $t \to \infty$, and find the limiting value.
Hint: Consider the general solution, equation (47). Show that the first term in the solution (47) is indeterminate with form $0 \cdot \infty$. Then, use l'Hôpital's rule to compute the limit as $t \to \infty$.

23. Show that if a and λ are positive constants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda t}$$

has the property that $y \to 0$ as $t \to \infty$.
Hint: Consider the cases $a = \lambda$ and $a \neq \lambda$ separately.

In each of Problems **24** through **27**, construct a first-order linear differential equation whose solutions have the required behavior as $t \to \infty$. Then solve your equation and confirm that the solutions do indeed have the specified property.

24. All solutions have the limit 3 as $t \to \infty$.

25. All solutions are asymptotic to the line $y = 3 - t$ as $t \to \infty$.

26. All solutions are asymptotic to the line $y = 2t - 5$ as $t \to \infty$.

27. All solutions approach the curve $y = 4 - t^2$ as $t \to \infty$.

28. **Variation of Parameters.** Consider the following method of solving the general linear equation of first order:

$$y' + p(t)y = g(t). \qquad \textbf{(48)}$$

a. If $g(t) = 0$ for all t, show that the solution is

$$y = A\exp\left(-\int p(t)\,dt\right), \qquad \textbf{(49)}$$

where A is a constant.

b. If $g(t)$ is not everywhere zero, assume that the solution of equation (48) is of the form

$$y = A(t)\exp\left(-\int p(t)\,dt\right), \qquad \textbf{(50)}$$

where A is now a function of t. By substituting for y in the given differential equation, show that $A(t)$ must satisfy the condition

$$A'(t) = g(t)\exp\left(\int p(t)\,dt\right). \qquad \textbf{(51)}$$

c. Find $A(t)$ from equation (51). Then substitute for $A(t)$ in equation (50) and determine y. Verify that the solution obtained in this manner agrees with that of equation (33) in the text. This technique is known as the method of **variation of parameters**; it is discussed in detail in Section 3.6 in connection with second-order linear equations.

In each of Problems **29** and **30**, use the method of Problem **28** to solve the given differential equation.

29. $y' - 2y = t^2e^{2t}$

30. $y' + \frac{1}{t}y = \cos(2t), \quad t > 0$

2.2 Separable Differential Equations

In Section 1.2 we used a process of direct integration to solve first-order linear differential equations of the form

$$\frac{dy}{dt} = ay + b, \qquad \textbf{(1)}$$

where a and b are constants. We will now show that this process is actually applicable to a much larger class of nonlinear differential equations.

We will use x, rather than t, to denote the independent variable in this section for two reasons. In the first place, different letters are frequently used for the variables in a differential equation, and you should not become too accustomed to using a single pair. In particular, x often occurs as the independent variable. Further, we want to reserve t for another purpose later in the section.

The general first-order differential equation is

$$\frac{dy}{dx} = f(x, y). \tag{2}$$

Linear differential equations were considered in the preceding section, but if equation (2) is nonlinear, then there is no universally applicable method for solving the equation. Here, we consider a subclass of first-order equations that can be solved by direct integration.

To identify this class of equations, we first rewrite equation (2) in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0. \tag{3}$$

It is always possible to do this by setting $M(x, y) = -f(x, y)$ and $N(x, y) = 1$, but there may be other ways as well. When M is a function of x only and N is a function of y only, then equation (3) becomes

$$M(x) + N(y)\frac{dy}{dx} = 0. \tag{4}$$

Such an equation is said to be **separable**, because if it is written in the **differential form**

$$M(x)\, dx + N(y)\, dy = 0, \tag{5}$$

then, if you wish, terms involving each variable may be placed on opposite sides of the equation. The differential form (5) is also more symmetric and tends to suppress the distinction between independent and dependent variables.

A separable equation can be solved by integrating the functions M and N. We illustrate the process by an example and then discuss it in general for equation (4).

EXAMPLE 2.2.1

Show that the equation

$$\frac{dy}{dx} = \frac{x^2}{1 - y^2} \tag{6}$$

is separable, and then find an equation for its integral curves.

Solution If we write equation (6) as

$$-x^2 + (1 - y^2)\,\frac{dy}{dx} = 0, \tag{7}$$

then it has the form (4) and is therefore separable. Recall from calculus that if y is a function of x, then by the chain rule,

$$\frac{d}{dx} f(y) = \frac{d}{dy} f(y)\,\frac{dy}{dx} = f'(y)\frac{dy}{dx}.$$

For example, if $f(y) = y - y^3/3$, then

$$\frac{d}{dx}\left(y - \frac{y^3}{3}\right) = (1 - y^2)\,\frac{dy}{dx}.$$

Thus the second term in equation (7) is the derivative with respect to x of $y - y^3/3$, and the first term is the derivative of $-x^3/3$. Thus equation (7) can be written as

$$\frac{d}{dx}\left(-\frac{x^3}{3}\right) + \frac{d}{dx}\left(y - \frac{y^3}{3}\right) = 0,$$

or

$$\frac{d}{dx}\left(-\frac{x^3}{3} + y - \frac{y^3}{3}\right) = 0.$$

Therefore, by integrating (and multiplying the result by 3), we obtain

$$-x^3 + 3y - y^3 = c, \tag{8}$$

where c is an arbitrary constant.

Equation (8) is an equation for the integral curves of equation (6). A direction field and several integral curves are shown in Figure 2.2.1. Any differentiable function $y = \phi(x)$ that satisfies equation (8) is a solution of equation (6). An equation of the integral curve passing through a particular point (x_0, y_0) can be found by substituting x_0 and y_0 for x and y, respectively, in equation (8) and determining the corresponding value of c.

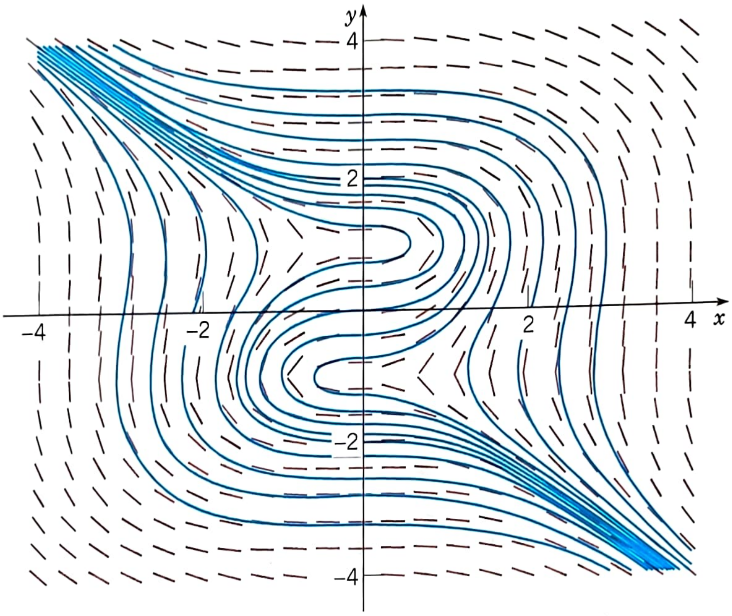

FIGURE 2.2.1 Direction field and integral curves of $y' = x^2/(1 - y^2)$.

Essentially the same procedure can be followed for any separable equation. Returning to equation (4), let H_1 and H_2 be any antiderivatives of M and N, respectively. Thus

$$H_1'(x) = M(x), \qquad H_2'(y) = N(y), \tag{9}$$

and equation (4) becomes

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0. \tag{10}$$

If y is regarded as a function of x, then according to the chain rule,

$$H_2'(y)\frac{dy}{dx} = \frac{d}{dy}H_2(y)\,\frac{dy}{dx} = \frac{d}{dx}H_2(y). \tag{11}$$

Consequently, we can write equation (10) as

$$\frac{d}{dx}(H_1(x) + H_2(y)) = 0. \tag{12}$$

By integrating equation (12) with respect to x, we obtain

$$H_1(x) + H_2(y) = c, \tag{13}$$

where c is an arbitrary constant. Any differentiable function $y = \phi(x)$ that satisfies equation (13) is a solution of equation (4); in other words, equation (13) defines the solution implicitly rather than explicitly. In practice, equation (13) is usually obtained from equation (5) by integrating the first term with respect to x and the second term with respect to y. The justification for this is the argument that we have just given.

The differential equation (4), together with an initial condition

$$y(x_0) = y_0, \tag{14}$$

forms an initial value problem. To solve this initial value problem, we must determine the appropriate value for the constant c in equation (13). We do this by setting $x = x_0$ and $y = y_0$ in equation (13) with the result that

$$c = H_1(x_0) + H_2(y_0). \tag{15}$$

Substituting this value of c in equation (13) and noting that

$$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)ds, \qquad H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)ds,$$

we obtain

$$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0. \tag{16}$$

Equation (16) is an implicit representation of the solution of the differential equation (4) that also satisfies the initial condition (14). Bear in mind that to determine an explicit formula for the solution, you need to solve equation (16) for y as a function of x. Unfortunately, it is often impossible to do this analytically; in such cases you can resort to numerical methods to find approximate values of y for given values of x.

EXAMPLE 2.2.2

Solve the initial value problem

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1, \tag{17}$$

and determine the interval in which the solution exists.

Solution The differential equation can be written as

$$2(y-1)dy = (3x^2 + 4x + 2)dx.$$

Integrating the left-hand side with respect to y and the right-hand side with respect to x gives

$$y^2 - 2y = x^3 + 2x^2 + 2x + c, \tag{18}$$

where c is an arbitrary constant. To determine the solution satisfying the prescribed initial condition, we substitute $x = 0$ and $y = -1$ in equation (18), obtaining $c = 3$. Hence the solution of the initial value problem is given implicitly by

$$y^2 - 2y = x^3 + 2x^2 + 2x + 3. \tag{19}$$

To obtain the solution explicitly, we must solve equation (19) for y in terms of x. That is a simple matter in this case, since equation (19) is quadratic in y, and we obtain

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}. \tag{20}$$

Equation (20) gives two solutions of the differential equation, only one of which, however, satisfies the given initial condition. This is the solution corresponding to the minus sign in equation (20), so we finally obtain

$$y = \phi(x) = 1 - \sqrt{x^3 + 2x^2 + 2x + 4} \tag{21}$$

as the solution of the initial value problem (15). Note that if we choose the plus sign by mistake in equation (20), then we obtain the solution of the same differential equation that satisfies the initial condition $y(0) = 3$. Finally, to determine the interval in which the solution (21) is valid, we must find the interval in which the quantity under the radical is positive. The only real zero of this expression is $x = -2$, so the desired interval is $x > -2$. Some integral curves of the differential equation are shown in Figure 2.2.2. The green curve passes through the point $(0, -1)$ and thus is the solution of the initial value problem (15). Observe that the boundary of the interval of validity of the solution (21) is determined by the point $(-2, 1)$ at which the tangent line is vertical.

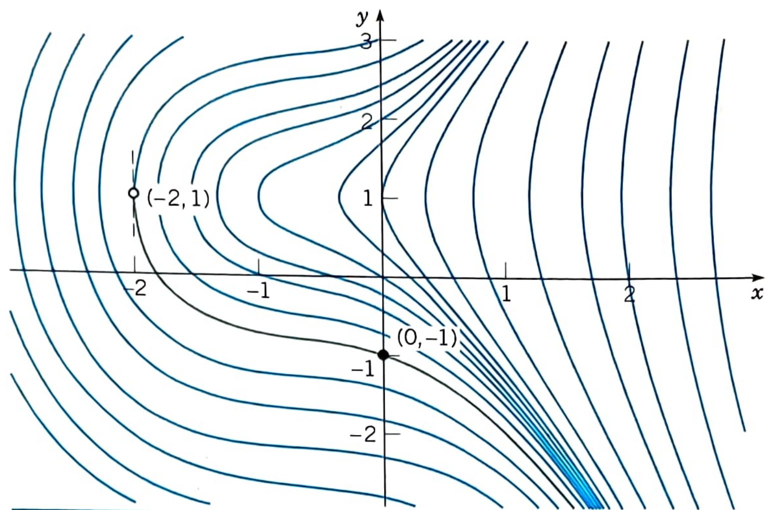

FIGURE 2.2.2 Integral curves of $y' = (3x^2 + 4x + 2)/2(y - 1)$; the solution satisfying $y(0) = -1$ is shown in green and is valid for $x > -2$.

EXAMPLE 2.2.3

Solve the separable differential equation

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3} \tag{22}$$

and draw graphs of several integral curves. Also find the solution passing through the point $(0, 1)$ and determine its interval of validity.

Solution Rewriting equation (22) as

$$(4 + y^3)\, dy = (4x - x^3)\, dx,$$

integrating each side, multiplying by 4, and rearranging the terms, we obtain

$$y^4 + 16y + x^4 - 8x^2 = c, \tag{23}$$

where c is an arbitrary constant. Any differentiable function $y = \phi(x)$ that satisfies equation (23) is a solution of the differential equation (22). Graphs of equation (23) for several values of c are shown in Figure 2.2.3.

To find the particular solution passing through $(0, 1)$, we set $x = 0$ and $y = 1$ in equation (23) with the result that $c = 17$. Thus the solution in question is given implicitly by

$$y^4 + 16y + x^4 - 8x^2 = 17. \tag{24}$$

It is shown by the green curve in Figure 2.2.3. The interval of validity of this solution extends on either side of the initial point as long as the function remains differentiable. From the figure we see that the interval ends when we reach points where the tangent line is vertical. It follows from the differential equation (22) that these are points where $4 + y^3 = 0$, or $y = (-4)^{1/3} \cong -1.5874$. From equation (24) the corresponding values of x are $x \cong \pm 3.3488$. These points are marked on the graph in Figure 2.2.3.

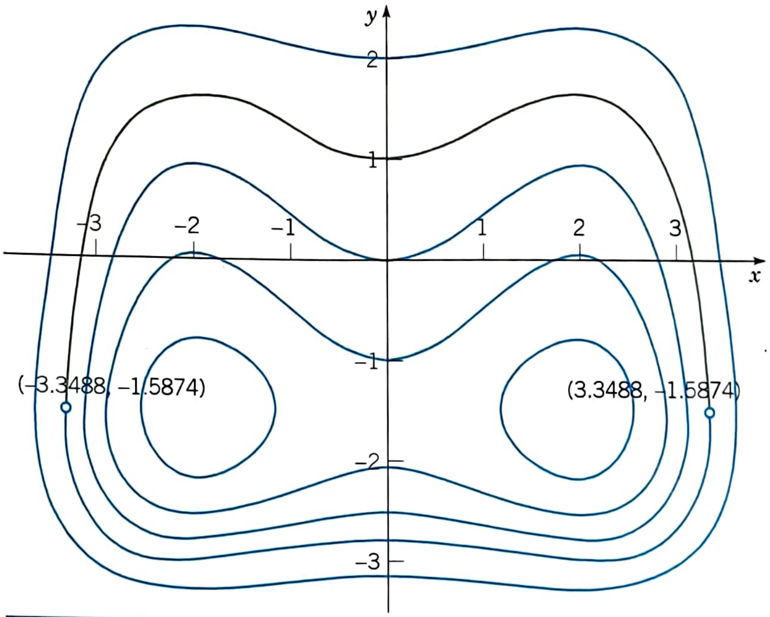

FIGURE 2.2.3 Integral curves of $y' = (4x - x^3)/(4 + y^3)$. The solution passing through $(0, 1)$ is shown by the green curve.

Note 1: Sometimes a differential equation of the form (2):

$$\frac{dy}{dx} = f(x, y)$$

has a constant solution $y = y_0$. Such a solution is usually easy to find because if $f(x, y_0) = 0$ for some value y_0 and for all x, then the constant function $y = y_0$ is a solution of the differential equation (2). For example, the equation

$$\frac{dy}{dx} = \frac{(y-3)\cos(x)}{1+2y^2} \tag{25}$$

has the constant solution $y = 3$. Other solutions of this equation can be found by separating the variables and integrating.

Note 2: The investigation of a first-order nonlinear differential equation can sometimes be facilitated by regarding both x and y as functions of a third variable t. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{26}$$

If the differential equation is

$$\frac{dy}{dx} = \frac{F(x, y)}{G(x, y)}, \tag{27}$$

then, by comparing numerators and denominators in equations (26) and (27), we obtain the system

$$\frac{dx}{dt} = G(x, y), \qquad \frac{dy}{dt} = F(x, y). \tag{28}$$

At first sight it may seem unlikely that a problem will be simplified by replacing a single equation by a pair of equations, but in fact, the system (28) may well be more amenable to investigation than the single equation (27). Chapter 9 is devoted to nonlinear systems of the form (28).

Note 3: In Example 2.2.2 it was not difficult to solve the initial value problem explicitly for y as a function of x. However, this situation is exceptional, and often it will be better to leave the solution in implicit form, as in Examples 2.2.1 and 2.2.3. Thus, in the problems below and in other sections where nonlinear equations appear, the words "solve the following differential equation" mean to find the solution explicitly if it is convenient to do so, but otherwise to find an equation defining the solution implicitly.

Problems

In each of Problems 1 through 8, solve the given differential equation.

1. $y' = \frac{x^2}{y}$
2. $y' + y^2 \sin(x) = 0$
3. $y' = \cos^2(x)\cos^2(2y)$
4. $xy' = (1 - y^2)^{1/2}$
5. $\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}$
6. $\frac{dy}{dx} = \frac{x^2}{1 + y^2}$
7. $\frac{dy}{dx} = \frac{y}{x}$
8. $\frac{dy}{dx} = \frac{-x}{y}$

In each of Problems 9 through 16:

a. Find the solution of the given initial value problem in explicit form.

b. G Plot the graph of the solution.

c. Determine (at least approximately) the interval in which the solution is defined.

9. $y' = (1 - 2x)y^2, \quad y(0) = -1/6$
10. $y' = (1 - 2x)/y, \quad y(1) = -2$
11. $x\,dx + ye^{-x}dy = 0, \quad y(0) = 1$
12. $dr/d\theta = r^2/\theta, \quad r(1) = 2$
13. $y' = xy^3(1 + x^2)^{-1/2}, \quad y(0) = 1$
14. $y' = 2x/(1 + 2y), \quad y(2) = 0$
15. $y' = (3x^2 - e^x)/(2y - 5), \quad y(0) = 1$
16. $\sin(2x)\,dx + \cos(3y)\,dy = 0, \quad y(\pi/2) = \pi/3$

Some of the results requested in Problems 17 through 22 can be obtained either by solving the given equations analytically or by plotting numerically generated approximations to the solutions. Try to form an opinion about the advantages and disadvantages of each approach.

17. G Solve the initial value problem

$$y' = \frac{1 + 3x^2}{3y^2 - 6y}, \quad y(0) = 1$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

18. G Solve the initial value problem

$$y' = \frac{3x^2}{3y^2 - 4}, \quad y(1) = 0$$

and determine the interval in which the solution is valid.
Hint: To find the interval of definition, look for points where the integral curve has a vertical tangent.

19. G Solve the initial value problem

$$y' = 2y^2 + xy^2, \quad y(0) = 1$$

and determine where the solution attains its minimum value.

20. G Solve the initial value problem

$$y' = \frac{2 - e^x}{3 + 2y}, \quad y(0) = 0$$

and determine where the solution attains its maximum value.

21. G Consider the initial value problem

$$y' = \frac{ty(4 - y)}{3}, \quad y(0) = y_0.$$

a. Determine how the behavior of the solution as t increases depends on the initial value y_0.

b. Suppose that $y_0 = 0.5$. Find the time T at which the solution first reaches the value 3.98.

22. G Consider the initial value problem

$$y' = \frac{ty(4 - y)}{1 + t}, \quad y(0) = y_0 > 0.$$

a. Determine how the solution behaves as $t \to \infty$.

b. If $y_0 = 2$, find the time T at which the solution first reaches the value 3.99.

c. Find the range of initial values for which the solution lies in the interval $3.99 < y < 4.01$ by the time $t = 2$.

23. Solve the equation

$$\frac{dy}{dx} = \frac{ay + b}{cy + d},$$

where $a, b, c,$ and d are constants.

24. Use separation of variables to solve the differential equation

$$\frac{dQ}{dt} = r(a + bQ), \quad Q(0) = Q_0,$$

where a, b, r, and Q_0 are constants. Determine how the solution behaves as $t \to \infty$

Homogeneous Equations. If the right-hand side of the equation $dy/dx = f(x, y)$ can be expressed as a function of the ratio y/x only, then the equation is said to be homogeneous.[1] Such equations can always be transformed into separable equations by a change of the dependent variable. Problem 25 illustrates how to solve first-order homogeneous equations.

25. N Consider the equation

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}. \tag{29}$$

a. Show that equation (29) can be rewritten as

$$\frac{dy}{dx} = \frac{(y/x) - 4}{1 - (y/x)}; \tag{30}$$

thus equation (29) is homogeneous.

[1] The word "homogeneous" has different meanings in different mathematical contexts. The homogeneous equations considered here have nothing to do with the homogeneous equations that will occur in Chapter 3 and elsewhere.

b. Introduce a new dependent variable v so that $v = y/x$, or $y = xv(x)$. Express dy/dx in terms of x, v, and dv/dx.

c. Replace y and dy/dx in equation (30) by the expressions from part **b** that involve v and dv/dx. Show that the resulting differential equation is

$$v + x\frac{dv}{dx} = \frac{v-4}{1-v},$$

or

$$x\frac{dv}{dx} = \frac{v^2-4}{1-v}. \qquad (31)$$

Observe that equation (31) is separable.

d. Solve equation (31), obtaining v implicitly in terms of x.

e. Find the solution of equation (29) by replacing v by y/x in the solution in part **d**.

f. Draw a direction field and some integral curves for equation (29). Recall that the right-hand side of equation (29) actually depends only on the ratio y/x. This means that integral curves have the same slope at all points on any given straight line through the origin, although the slope changes from one line to another. Therefore, the direction field and the integral curves are symmetric with respect to the origin. Is this symmetry property evident from your plot?

The method outlined in Problem 25 can be used for any homogeneous equation. That is, the substitution $y = xv(x)$ transforms a homogeneous equation into a separable equation. The latter equation can be solved by direct integration, and then replacing v by y/x gives the solution to the original equation. In each of Problems 26 through 31:

a. Show that the given equation is homogeneous.

b. Solve the differential equation.

c. **G** Draw a direction field and some integral curves. Are they symmetric with respect to the origin?

26. $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$

27. $\dfrac{dy}{dx} = \dfrac{x^2 + 3y^2}{2xy}$

28. $\dfrac{dy}{dx} = \dfrac{4y - 3x}{2x - y}$

29. $\dfrac{dy}{dx} = -\dfrac{4x + 3y}{2x + y}$

30. $\dfrac{dy}{dx} = \dfrac{x^2 - 3y^2}{2xy}$

31. $\dfrac{dy}{dx} = \dfrac{3y^2 - x^2}{2xy}$

2.3 Modeling with First-Order Differential Equations

Differential equations are of interest to nonmathematicians primarily because of the possibility of using them to investigate a wide variety of problems in the physical, biological, and social sciences. One reason for this is that mathematical models and their solutions lead to equations relating the variables and parameters in the problem. These equations often enable you to make predictions about how the natural process will behave in various circumstances. It is often easy to vary parameters in the mathematical model over wide ranges, whereas this may be very time-consuming or expensive, if not impossible, in an experimental setting. Nevertheless, mathematical modeling and experiment or observation are both critically important and have somewhat complementary roles in scientific investigations. Mathematical models are validated by comparison of their predictions with experimental results. On the other hand, mathematical analyses may suggest the most promising directions to explore experimentally, and they may indicate fairly precisely what experimental data will be most helpful.

In Sections 1.1 and 1.2 we formulated and investigated a few simple mathematical models. We begin by recapitulating and expanding on some of the conclusions reached in those sections. Regardless of the specific field of application, there are three identifiable steps that are always present in the process of mathematical modeling.

Step 1: Construction of the Model. In this step the physical situation is translated into mathematical terms, often using the steps listed at the end of Section 1.1. Perhaps most critical at this stage is to state clearly the physical principle(s) that are believed to govern the process. For example, it has been observed that in some circumstances heat passes from a warmer to a cooler body at a rate proportional to the temperature difference, that objects move about in accordance with Newton's laws of motion, and that isolated insect populations grow at a rate proportional to the current population. Each of these statements involves a rate of change (derivative) and consequently, when expressed mathematically, leads to a differential equation. The differential equation is a mathematical model of the process.