Groupoids and Property (T)

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Kazhdan's Property (T)

Let G be a topological group. A **unitary representation** of G is a pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a Hilbert space and π is a family of unitary operators $\pi_g : \mathcal{H} \to \mathcal{H}$ indexed by G, satisfying:

(i) (identity) $\pi_e = I_{\mathcal{H}}$.

(ii) (compatibility) $\pi_g \pi_h = \pi_{gh}$ for all $g, h \in G$.

Also, the map

$$\begin{aligned} G \times \mathcal{H} &\to \mathcal{H} \\ (g, \xi) &\mapsto \pi_g \xi \end{aligned}$$

is continuous.

- Let $(\mathcal{H}, \pi)$ be a representation of G. A **subrepresentation** of $(\mathcal{H}, \pi)$ is a closed G-invariant subspace.
- Two representations $(\mathcal{H}, \pi)$ and $(\mathcal{K}, \rho)$ are **unitarily equivalent** if there exists a unitary operator $T : \mathcal{H} \to \mathcal{K}$ such that the the following diagram commutes for all $g \in G$,

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\pi_g"] \arrow[d, "T"'] & \mathcal{H} \arrow[d, "T"] \\ \mathcal{K} \arrow[r, "\rho_g"'] & \mathcal{K} \end{tikzcd}$$

- A representation π is **(strongly) contained** in ρ, and write

$$\pi \subset \rho$$

if π is unitarily equivalent to a subrepresentation of ρ.

- In the 1948 Godement introduced the notion of **weak containment** of representations, which was further developed by Fell in the early 1960s.
- Let G be a topological group. Fix a representation $(\mathcal{H}, \pi)$ and a vector $\xi \in \mathcal{H}$. The continuous function $\phi : G \to \mathbb{C}$ defined by

$$\phi(g) = \langle \pi_g \xi, \xi \rangle$$

  is called a **positive type function associated to** π.
- A positive type function ϕ is **normalized** if $\phi(e) = 1$.
- The collection of normalized positive type functions on G is denoted $\mathcal{P}_1(G)$.

Kazhdan's Property (T)

- Let $(\mathcal{H}, \pi)$ and $(\mathcal{K}, \rho)$ be two representations of G. We say π is **weakly contained** in ρ, and write

$$\pi \prec \rho$$

  if any function in $\mathcal{P}_1(G)$ associated to π can be approximated uniformly on compact subets of G by *convex combinations* of functions in $\mathcal{P}_1(G)$ associated to ρ.
- In symbols, for any unit vector $\xi \in \mathcal{H}$, any compact subset $Q \subset G$, and any $\epsilon > 0$, there exist unit vectors $\eta_1, \ldots, \eta_n \in \mathcal{K}$ and constants $0 \leq t_1, \ldots, t_n \leq 1$ with $\sum t_i = 1$ such that

$$\sup_{g \in Q} |\langle \pi_g \xi, \xi \rangle - \sum_{i=1}^{n} t_i \langle \rho_g \eta_i, \eta_i \rangle| < \epsilon.$$

Kazhdan's Property (T)

- The representation $(\mathbb{C}, \pi_0)$ where

$$\pi_0(g) : \mathbb{C} \to \mathbb{C}$$
$$\pi_0(g)(z) = z$$

  for all $g \in G$ is called the **trivial representation**.
- A representation is **irreducible** if it contains no nontrivial subrepresentations. For example, π_0 is irreducible.
- The collection of all irreducible representations of a topological group G is denoted $\widehat{G}$.
- $\widehat{G}$ is a topological space; its topology was studied by Fell in the early 1960s.

- A topological group G has **property (T)** if $\{\pi_0\}$ is an open set in $\widehat{G}$.
- Property (T) was introduced by Kazhdan in 1967.
- Kazhdan was attempting to demonstrate certain lattices were finitely generated.
  Recall: a subgroup Γ of locally compact group G is a lattice if Γ is discrete and G/Γ carries a G-invariant probability measure.
- Example with property (T): any compact group
- Example without property (T): $\mathbb{Z}$

- Property (T) is equivalent to the condition:

$$\forall \pi \; (\pi_0 \prec \pi \Rightarrow \pi_0 \subset \pi)$$

- A representation $(\mathcal{H}, \pi)$ has **almost invariant vectors** if for any compact subset $Q \subset G$ and every $\epsilon > 0$ there exists a unit vector $\xi \in \mathcal{H}$ such that

$$\sup_{g \in Q} \|\pi_g \xi - \xi\| < \epsilon.$$

Kazhdan's Property (T)

- $\pi_0 \prec \pi$ if and only if π has almost invariant vectors.
- $\pi_0 \subset \pi$ if and only if π has a nonzero invariant vector η, i.e.

$$\forall g \in G, \ \pi_g \eta = \eta.$$

- Hence, property (T) is equivalent to the condition: any representation that has almost invariant vectors has an invariant vector.

Definition

A **groupoid** is a small category with inverses.

$$G := \text{set of arrows (the } \textbf{groupoid}\text{)}$$
$$G^{(0)} := \text{set of objects (the } \textbf{base space}\text{)}$$

- The elements of $G^{(0)}$ are called **units**.
- The functions $r, s : G \to G^{(0)}$ denote the the **range** and **source** maps. If $g : x \to y$ is an arrow, then

$$s(g) = x \text{ and } r(g) = y.$$

Groupoids

$$\begin{aligned} G^{(2)} &:= \{(g,h) : s(g) = r(h)\} && \text{(composable pairs)} \\ m &: G^{(2)} \to G && \text{(composition)} \\ \iota &: G \to G && \text{(inversion)} \end{aligned}$$

A **topological groupoid** is a groupoid equipped with a topology making m and ι continuous.

Groupoids

- The **range fiber** at $x \in G^{(0)}$ is the set
$$G^x := \{g \in G : r(g) = x\}.$$
- The **isotropy group** at $x \in G^{(0)}$ is the group
$$G|_x := \{g \in G : s(g) = r(g) = x\}.$$
- Examples of groupoids: groups, group bundles, equivalence relations, group actions.

Example: Let X be a topological space. The *fundamental groupoid* $G = \Pi_1(X)$ of X:

- The base space $G^{(0)} = X$.
- The arrows are all homotopy classes of endpoint preserving paths in X.
- Each $G|_x$ is the fundamental group $\pi_1(X, x)$ with basepoint x.

Let G be a locally compact groupoid. A **Haar system** is a family of Radon measures $\lambda = \{\lambda^x : x \in G^{(0)}\}$ on G which satisfy:

(i) Each λ^x only "sees" G^x.

(ii) The measures are translation invariant in a suitable sense.

A Haar system plays the role of a Haar measure on a locally compact group.

Continuous Banach Bundles

A **continuous Banach bundle** over the space X is a continuous open surjection $\pi : E \to X$ where each fiber $E_x := \pi^{-1}(x)$ is a Banach space and the "induced" structure maps

$$u \mapsto \lambda u$$
$$(u, v) \mapsto u +_{E_x} v$$
$$u \mapsto \|u\|_{E_x}$$

are all continuous where they make sense.

Continuous Banach Bundles

- Example: to each second countable locally compact groupoid G with Haar system λ and $1 \leq p < \infty$ we can associate a unique continuous Banach bundle over $G^{(0)}$,

$$L^p(G, \lambda) := \bigsqcup_{x \in G^{(0)}} L^p(\lambda^x).$$

- A **continuous Hilbert bundle** is a continuous Banach bundle where every fiber is a Hilbert space.

Continuous Banach Bundles

Let E and E' be continuous Banach bundles over X.

- A **section** or **vector field** is a function $\xi : X \to E$ where $\xi_x \in E_x$ for each $x \in X$.
- The space of continuous sections is denoted $C(X, E)$.
- The **dual** of E, denoted E^*, is the collection of all continuous maps $\phi : E \to \mathbb{C}$ such that the restriction $\phi_x := \phi|E_x$ is linear for all $x \in X$.
- We call the members of E^* **functionals**.

Continuous Banach Bundles

Definition (C.)

Let E be a continuous Banach bundle over X. The **weak* topology** on E^* is defined by:

$$\phi_i \to \phi \text{ in } E^* :\Leftrightarrow \forall f \in E,\ \phi_i(f) \to \phi(f) \text{ in } \mathbb{C}.$$

The weak* topology is the topology of pointwise convergence.

Theorem (C.)

Let E be a continuous Banach bundle over X. Then the set

$$(E^*)_1 := \{\phi \in E^* : \|\phi_x\|_{E_x^*} \leq 1 \textit{ for all } x \in X\}$$

is weak compact.*

- Extension of Banach-Alaoglu theorem.
- For lack of a better term we will call $(E^*)_1$ the *unit tube*.

Let G be a topological groupoid. A **unitary representation** of G is a pair (E, L) where E is a continuous Hilbert bundle over $G^{(0)}$ and L is a family of unitary operators $L_g : E_{s(g)} \to E_{r(g)}$ indexed by G satisfying:

(i) (identity) $L_x = I_x$ for each $x \in G^{(0)}$.

(ii) (compatibility) $L_g L_h = L_{gh}$ for each $(g, h) \in G^{(2)}$.

Also, the map

$$\begin{aligned} \{(g, u) : G \times E : u \in E_{s(g)}\} &\to E \\ (g, u) &\mapsto L_g u \end{aligned}$$

is continuous.

Continuous Unitary Representations

- Let (E, L) be a representation of a topological groupoid. A **subrepresentation** is a closed G-invariant subbundle.
- A **unitary equivalence** between representations (E, L) and (E', L') is an isometric isomorphism $T : E \to E'$ where the following diagram commutes for all $g \in G$,

$$\begin{array}{ccc} E_{s(g)} & \xrightarrow{L_g} & E_{r(g)} \\ {\scriptstyle T_{s(g)}}\downarrow & & \downarrow{\scriptstyle T_{r(g)}} \\ E'_{s(g)} & \xrightarrow[L'_g]{} & E'_{r(g)} \end{array}$$

- (E', L') **(strongly) contains** (E, L), denoted $L \subset L'$, if (E, L) is unitarily equivalent to a subrepresentation of (E', L').

Let G be a topological groupoid.

- Let G be a topological groupoid. Fix a representation (E, L) and a section $\xi \in C(G^{(0)}, E)$. The continuous function $\phi : G \to \mathbb{C}$ defined by

$$\phi(g) = \langle \xi_{r(g)}, L_g \xi_{s(g)} \rangle_{E_{r(g)}},$$

  is a **positive type function associated** to L.
- A positive type function ϕ is **normalized** if

$$\phi(x) = 1 \text{ for all } x \in G^{(0)}$$

- The collection of normalized positive type functions is denoted $\mathcal{P}_1(G)$.

Continuous Unitary Representations

Theorem (C.)

Let G be a second countable locally compact groupoid with Haar system λ. There is an embedding $\mathcal{P}_1(G) \to L^1(G, \lambda)^*$.

- The space $\mathcal{P}_1(G)$ can be viewed as a subset of the locally convex topological vector space $L^1(G, \lambda)^*$.
- $\mathcal{P}_1(G)$ can be endowed with the weak* subspace topology.

Continuous Unitary Representations

Definition (C.)

The **fiberwise compact convergence (f.c.c.) topology** on $\mathcal{P}_1(G)$ is the topology where $\phi_i \to \phi$ in $\mathcal{P}_1(G)$ if and only if

$$\forall x \in G^{(0)},\ \phi_i|G^x \to \phi|G^x \text{ uniformly on compact sets in } G^x.$$

Theorem (C.)

Let G be a second countable locally compact groupoid with Haar system. The weak and f.c.c. topologies on $\mathcal{P}_1(G)$ coincide.*

- Extension of Raikov's observation through an adaptation of Yoshizawa's argument.

Continuous Unitary Representations

Definition (Bos)

A unitary representation (E, L) of G is $G^{(0)}$-**irreducible** if the restriction of L to each isotropy group is an irreducible group representation.

Example: the representation $(G^{(0)} \times \mathbb{C}, L_0)$ where G acts trivially, called the 1-dimensional **trivial representation**, is $G^{(0)}$-irreducible.

Theorem (C.)

Let $\phi \in \mathcal{P}_1(G)$ be associated to the representation (E, L) of G. If (E, L) is $G^{(0)}$-irreducible, then ϕ is an extreme point in $\mathcal{P}_1(G)$.

Weak Containment

Definition (C.)

Let (E, L) and (E', L') be two representations of G. We say (E, L) is **weakly contained** in (E', L'), and write

$$L \prec L'$$

if any function in $\mathcal{P}_1(G)$ associated to L can be approximated f.c.c. by *convex combinations* of functions in $\mathcal{P}_1(G)$ associated to L'.

Theorem (C.)

Let G be a second countable locally compact groupoid with Haar system. Suppose $L \prec L'$ and L is $G^{(0)}$-irreducible. Then every $\phi \in \mathcal{P}_1(G)$ associated to L can be approximated f.c.c. by functions in $\mathcal{P}_1(G)$ associated to L'.

A consequence of the Krein-Millman theorem: Let X be a locally convex topological vector space and let $A \subset X$. Suppose $C = \overline{\mathrm{conv}(A)}$ is compact. Then

$$\mathrm{ext}(C) \subset \overline{A}.$$

Sketch:

$$X := L^1(G, \lambda)^*$$
$A :=$ all functions in $\mathcal{P}_1(G)$ associated to L'
$C := \overline{\mathrm{conv}(A)}^{weak*} \subset (L^1(G, \lambda)^*)_1$ is weak* compact
$G^{(0)}$-irreducibility and $L \prec L' \Rightarrow \phi \in \mathrm{ext}(C)$
$$\therefore \phi \in \overline{A}^{weak*} = \overline{A}^{f.c.c.}$$

Let (E, L) be a representation of G. A section $\xi : G^{(0)} \to E$ is **unital** if

$$\|\xi_x\| = 1 \text{ for all } x \in G^{(0)}$$

Definition (C.)

We say (E, L) has **almost invariant sections** if for any $x \in G^{(0)}$, $Q \subset G^x$ compact, and $\epsilon > 0$, there exists a unital section $\xi \in C(G^{(0)}, E)$ such that

$$\sup_{g \in Q} \|L_g \xi_{s(g)} - \xi_x\|_{E_x} < \epsilon.$$

A Definition of Property (T)

Theorem (C.)

Let (E, L) be a representation of a second countable locally compact groupoid with Haar system. Then the following are equivalent.

- $L_0 \prec L$.
- (E, L) *has almost invariant sections.*

Theorem (C.)

Let (E, L) be a representation of a topological groupoid G. Then the following are equivalent:

- $L_0 \subset L$.
- (E, L) *contains a continuous unital invariant section* η, *that is,* $L_g \eta_{s(g)} = \eta_{r(g)}$ *for all* $g \in G$.

A Definition of Property (T)

Definition (C.)

Let G be a topological groupoid. We say G has **property (T)** if any representation of G that has almost invariant sections has a continuous unital invariant section.

Examples of groupoids with property (T):

- Groups have Kazhdan's property (T) if and only if they have groupoid property (T).
- A groupoid is **transitive** if any two objects have an arrow between them. Transitive second countable locally compact fiberwise compact groupoids with open range and source maps have property (T).

Future Work

- Applications?
- Explore different notions of irreducibility. If $\phi \in \mathrm{ext}(\mathcal{P}_1(G))$, then L^ϕ is [*insert some notion of irreducibility*]. Or, weaken the notion of extreme point, $C(G^{(0)})$-modules.
- Explore property (T) for the measurable representation theory of a groupoid
- What is the relationship between weak containment and property (T) with the groupoid C^*-algebra $C^*(G)$.