MATH 611 **Due Wednesday, November 18, 1992**

1. **a)** Prove that if $\{L_i\}_{i\in I}$ is a family of left ideals in a ring R such that R is generated as an R-module by $\bigcup_I L_i$, then there is a **finite** subset $J \subseteq I$ such that R is generated by $\bigcup_J L_i$. (HINT: R is a cyclic R-module generated by 1.)
b) Prove that if a ring R has a minimal (proper) left ideal L which is not contained in any proper (two-sided) ideal, then R is a *finite* direct sum of left ideals isomorphic to L and R has no non-trivial proper ideals.

2. **a)** Let R_1 and R_2 be ideals in a ring R such that $R = R_1 \times R_2$. (Refer to problem **2** of the October 7 homework and also to pp. 130–131 of Hungerford.) Prove that there exist ideals $0 \neq I,\ J \subseteq R_1$ such that $R_1 = I \oplus J$ if and only if there exists an idempotent e in the center of R such that $0 \neq e \in R_1$ and $eR_1 \subsetneq R_1$.
b) Suppose that $R = R_1 \times \cdots \times R_s = P_1 \times \cdots \times P_t$, where the R_i and P_j are indecomposable as (two-sided) ideals (or, equivalently, do not contain any central idempotent except the one that generates them). Prove that $s = t$ and the P_j can be renumbered so that for all i, $R_i = P_i$.

Definition. An R-module M is called **simple** if $M \neq 0$ and M has no non-trivial proper submodules.

3. Prove that a left R-module is simple if and only if it is isomorphic to R/L, where L is a maximal left ideal.

Fact. It can be shown using Zorn's Lemma that if r is a non-invertible element in a ring then there exists a maximal left ideal containing r and a maximal right ideal containing r.

4. Let R be a ring and $r \in R$. Prove that the following conditions are equivalent and that the set of $r \in R$ satisying these conditions forms a (two-sided) ideal.

(1) $rM = 0$ for all simple left R-modules M.
(2) $Nr = 0$ for all simple right R-modules N.
(3) $\varphi(r) = 0$ for all homomorphisms φ from R into a simple left R-module.
(4) $\psi(r) = 0$ for all homomorphisms ψ from R into a simple right R-module.
(5) r belongs to all maximal left ideals in R.
(6) r belongs to all maximal right ideals in R.
(7) For all $x \in R$, $1 - xr$ is invertible.
(8) For all $x \in R$, $1 - rx$ is invertible.

5. Prove that if a left ideal L consists entirely of nilpotent elements then L is contained in every maximal left ideal.
WARNING: This does not mean that every nilpotent element in a ring belongs to all maximal left ideals.

ANSWER FOR NOVEMBER 18 HOMEWORK

1. **b)** If L is not contained in any proper (two-sided) ideal then the ideal generated by L must be R. This ideal is LR, which is the right ideal spanned by $\bigcup\{Lr \mid r \in R\}$. By part **a)**, then, there exist elements $r_1, \ldots, r_n \in R$ such that $R = Lr_1 + \cdots + Lr_n$. By omitting any superfluous elements r_i in this sum, we may suppose that for all i,

$$Lr_i \not\subseteq Lr_1 + \cdots + \widehat{Lr_i} + \cdots + Lr_n.$$

Now let $\varphi\colon L \to Lr_i$ be given by $\ell \mapsto \ell r_i$. Then φ is an epimorphism and $\operatorname{Ker}\varphi$ is a left ideal contained in L, so since L is a minimal left ideal either $\operatorname{Ker}\varphi = 0$ or $\operatorname{Ker}\varphi = L$. But $Lr_i \neq 0$ otherwise it could be omitted from the sum above. Thus φ is monic, so $Lr_i \approx L$. In particular, Lr_i is a simple left R-module . It then follows that

$$Lr_i \cap (Lr_1 + \cdots + \widehat{Lr_i} + \cdots + Lr_n) = 0. \qquad \text{(WHY?)}$$

Since this holds for all i it follows that $R = Lr_1 \oplus \cdots \oplus Lr_n$.
Now suppose that I is a non-trivial ideal in R. Note that since R is a finite direct sum of artinian modules (the Lr_i are certainly artinian, since they are simple), then R is artinian. It then follows that I must contain a minimal left ideal L'. Thus I must contain the ideal $L'R$ generated by L'. But as seen above, $L'R = R$. Thus $I = R$.